A Queue with Periodic Arrivals and Constant Service Rate

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ABSTRACT: Consider a queueing system in which K sources each generate $1/M$ units of work once every time unit. The phases of the sources are mutually independent, and each phase is uniformly distributed over the unit interval. The queue is served at unit rate. The distribution of the typical work and the maximum work are studied both for finite K and M and in the limit of large K and M. The limiting maximum work is identified as the maximum of a reflecting Brownian motion over the unit interval given that its local time at zero first reaches a specified value at the end of the interval.

The analysis is expedited by a tie to the empirical process arising in Kolmogorov-Smirnov statistical tests.

1 INTRODUCTION

Suppose the work arriving to a queueing system by time t is given by

$$A(t) = \frac{1}{M} \sum_{i=1}^{K} I_{\{\Theta_i \leq t\}}$$

for $0 \leq t \leq 1$, where M and K are positive integers, and $\Theta_1, \cdots, \Theta_K$ are independent and uniformly distributed on the interval [0,1]. Extend A to be defined for all $t \geq 0$ by requiring $A(t+1) = A(t) + \frac{K}{M}$ for all $t \geq 0$. The process A represents the work generated by K independent periodic sources, each generating $\frac{1}{M}$ units of work once every time unit, with uniform phases.

The work at time t in a system with constant service rate 1 and arrival process A is defined by

$$W(t) = \sup_{0 \leq s \leq t} \{A(t) - A(s) - (t - s)\}. \tag{1.1}$$

Suppose henceforth that $K \leq M$. Then $A(s) - A(s-1) - 1 \leq 0$ for all $s \geq 1$, so the maximization in (1.1) can be restricted to s in the interval $(t-1)_+ \leq s \leq t$ without changing the value of the maximum. Hence,

$$W(t) = \sup_{0 \leq u \leq 1} \{A(t) - A(t-u) - u\}$$

for $t \geq 1$. Consequently, the process $(W(t) : t \geq 1)$ has sample paths of period one, and the process is stationary since A has stationary increments. Define the random variable

$$\tilde{W} = \sup_{0 \leq u \leq 1} \{A(u) - u\}. \tag{1.2}$$

Then $\tilde{W}$ has the same distribution as $W(t)$ for any fixed $t \geq 1$.

The maximum work in the system $W_{\max}$ is given by

$$\begin{aligned} W_{\max} &= \max_{t \geq 0} W(t) = \max_{0 \leq t \leq 2} W(t) & (1.3) \\ &= \max_{0 \leq s \leq t \leq 2} \{A(t) - A(s) - (t-s)\} & (1.4) \end{aligned}$$

The purpose of this note is to examine the distributions of $\tilde{W}$ and $W_{\max}$, both for fixed K and M, and in the limit of large K and M.

The task is facilitated by a direct connection to mathematical statistics. Specifically,

$$A(t) = \frac{\sqrt{K}}{M} U_K(t) + \frac{Kt}{M}, \quad 0 \leq t \leq 1 \tag{1.5}$$

where U_K defined by

$$U_K(t) = \sqrt{K} \left\{ \frac{1}{K} \sum_{i=1}^{K} I_{\{\Theta_i \leq t\}} - t \right\}$$

is known as the empirical process with parameter K [18]. The empirical process is central to the theory of goodness of fit, being a normalized difference between the sample distribution function of $\Theta_1, \cdots, \Theta_K$, and the actual distribution function of these variables.

2 DISTRIBUTIONS FOR FINITE K AND M

The distribution of $\tilde{W}$ was found independently by Pyke [14] and Dempster [3], and Takacs [19] found an elegant derivation using the Ballot Theorem. The distribution is given by

$$P[\tilde{W} \leq c] \;=\; 1 - P[A(t) \geq t + c \;\; \text{for some} \;\; t]$$

$$= \quad 1 - \sum_{j=\lceil Mc\rceil}^{K} \binom{K}{j} (\frac{j}{M} - c)^j (1 - \frac{j}{M} + c)^{K-j} \left(\frac{M(1+c)-K}{M(1+c)-j}\right) \tag{2.1}$$

for $0 \le c \le K/M$. The jth term on the right is simply, by the Ballot Theorem, the probability that $A(t) = t + c$ for the last time at $t = \frac{j}{M} - c$.

Let $\lambda > 0$ and consider an $M/D/1$ queue with arrival rate λ and service times $1/M$. Let $r_c(i)$ denote the probability that during a typical busy period of the $M/D/1$ queue, exactly i customers arrive and are served, and the workload remains less than or equal to c throughout the busy period. Then, for $K < M$,

$$P\left[W_{\max} \le c\right] = \frac{\sum_{j=1}^{K} r_c^{*j}(K) poi(\lambda(1-\frac{K}{M}), j)}{poi(\lambda, K)(1-\frac{K}{M})} \tag{2.2}$$

where r_c^{*j} denotes the j-fold convolution of r_c, and $poi(\mu, i) = \exp(-\mu)\mu^i/i!$.

To verify (2.2), first observe that $P[W_{\max} \le c] = P[W_{\max} \le c | W(1) = 0]$, since $(W(t) : t \ge 1)$ is stationary and $\int_t^{t+1} I_{\{W(s)=0\}} ds = 1 - \frac{K}{M}$ for all $t \ge 1$ with probability one. Given $W(1) = 0$, $(W(t) : t \ge 0)$ is periodic so that $P[W_{\max} \le c] = P[\max_{0 \le t \le 1} W_t \le c | W(1) = 0]$. The process $[W_t : 0 \le t \le 1 | W(1) = 0]$ has the same distribution as the workload process in the $M/D/1$ queue conditioned on the following event: K customers arrive during the interval $[0,1]$ and the server is idle at time 1. The denominator of the righthand side of (2.2) is the total probability of this event.

While the $M/D/1$ queue is idle, busy periods of the queue are initiated at rate λ, so $poi(\lambda(1-\frac{K}{M}), j)$ is the probability exactly j busy periods occur when the cumulative idle time just reaches $1-\frac{K}{M}$. The factor $r_c^j(K)$ is the probability that during j busy periods a total of K customers arrive and the work remains less than or equal to c throughout the M busy periods.

In case $K = M$,

$$P[W_{\max} \le c] = \frac{r_c(K)\lambda}{poi(\lambda, K)}. \tag{2.3}$$

A proof of (2.3) which is similar to the proof of (2.2) can be fashioned. A quicker method is to note that (2.2) is valid even if M is not integer valued (in fact so is (2.1)), and then let M converge down to K for K fixed.

Equations (2.2) and (2.3) are valid for any $\lambda > 0$, though the choice $\lambda = K$ leads to good numerical conditioning.

To complete the computation of the distribution of $W_{\max}$, a method for computing the distribution r_c is needed. A recursive solution to this and more general problems is well known in the theory of empirical processes (see [18]). It is based on the following simple observation, which allows us to calculate the workload distribution during an $M/D/1$ busy period (defined in continous time) by only updating distributions at a sequence of times. Writing c as $c = \lfloor Mc \rfloor / M + \epsilon$ where $0 \leq \epsilon < 1/M$, given a busy period starts at time zero, the work in the $M/D/1$ queue stays in the interval $(0, c)$ as long as it does so at times of the form i/M or $i/M + \epsilon$.

Methods for computing the exact distributions of $\tilde{W}$ and $W_{\max}$ for a discrete time version of the queueing problem considered in this paper were given by Ott and Shantikumar [13]. In the discrete time case the distribution of $W_{\max}$ for $K < M$ is simpler than that given by (2.2): a single $M - K$ fold convolution of r_c is needed because, properly defined, there are exactly $M - K$ busy periods for the discrete time model. In contrast, the computation of $W_{\max}$ for $K = M$ is somewhat more complicated in discrete time than the corresponding expression (2.3) due to the fact that the queue length process can reach its minimum multiple times per period in discrete time. Related literature, including the calculation of blocking probability for finite buffer capacity, is cited in [15].

A brief history of the queueing model is as follows. Eckberg [6] introduced the model to describe the multiplexing of information sources which each generate a periodic stream of information packets, and he gave an algorithmic approach to compute the equilibrim distribution. Gravey [7] gave another algorithm and suggested the name N*D/D/1 queue. The expression (2.1) was introduced to to the literature on the model independently in [16] and [1].

3 ASYMPTOTIC DISTRIBUTION OF $\tilde{W}$

An asymptotic analysis of $\tilde{W}$ and $W_{\max}$ can be based on the fact that the empirical process U_K for large K closely approximates the Brownian bridge process. Indeed, Doob [5] conjectured and Donsker [4] proved that U_K converges weakly to the Brownian bridge. The continuous mapping theorem for weak convergence gives representations for limit random variables as functions of the

Brownian bridge.

The Brownian bridge process B_0 is a sample continuous Gaussian random process with mean zero and autocorrelation function $E[B_0(s)B_0(t)] = s(1-t)$ for $0 \leq s \leq t \leq 1$. Doob [5] showed that

$$P[\max_{0\leq t\leq 1} \{B_0(t) - \alpha t\} \leq \beta] = 1 - \exp(-2\beta(\alpha+\beta)). \tag{3.1}$$

He also noted that $((1+s)B_0(\frac{1}{1+s}) : s \geq 0)$ is a standard Brownian motion, as can be checked by calculating its autocovariance function, so that (3.1) is equivalent to the more widely known inequality for the standard Brownian motion B:

$$P[B(s) \leq (\alpha+\beta) + \beta s, \forall s \geq 0] = 1 - \exp(-2\beta(\alpha+\beta)). \tag{3.2}$$

A useful result about the error of approximating an empirical process by a Brownian bridge is given by Bretagnolle and Massart [2], and can be stated as follows. There exist constants c_1, c_2, c_3 (for example $c_1 = 12, c_2 = 2, c_3 = 1/6$) so that for each positive integer K, there is a construction of U_K and B_0 on a single probability space so that

$$P[\sup_{0\leq t\leq 1} |U_K(t) - B_0(t)| \geq \frac{x + c_1 \log K}{\sqrt{K}}] \leq c_2 \exp(-c_3 x). \tag{3.3}$$

We symbolically summarize this result by writing

$$U_K(t) = B_0(t) + \frac{O(\log K)}{\sqrt{K}}. \tag{3.4}$$

Combining (1.2), (1.5) and (3.4) yields

$$\begin{aligned} \tilde{W} &= \sup_{0\leq t\leq 1} \left\{ \frac{\sqrt{K}}{M}[B_0(t) + \frac{O(\log K)}{\sqrt{K}}] - \frac{(M-K)t}{M} \right\} && (3.5) \\ &= \frac{\sqrt{K}}{M} \max_{0\leq t\leq 1} \left\{ B_0(t) - \frac{M-K}{\sqrt{K}} t \right\} + \frac{O(\log K)}{M} && (3.6) \end{aligned}$$

where in the later expression $O(\log K)$ represents a random variable that exceeds $x + c_1 \log K$ with probability at most $c_2 \exp(-c_3 x)$.

Take $\alpha = \frac{M-K}{\sqrt{K}}$ in (3.1) and rescale B_0 by multiplying by $\frac{\sqrt{K}}{M}$ to deduce from (3.5) that

$$P[\tilde{W} \leq c + \frac{O(\log K)}{M}] = 1 - \exp(-2cM(M + Mc - K)/K). \tag{3.7}$$

Finally, given u with $0 < u \leq 1$, solve for c so that the righthand side of (3.7) is equal to u to yield that $P[\tilde{W} \leq q(M,K,u) + \frac{O(\log K)}{M}] = u$ where

$$q(M,K,u) = \frac{\sqrt{(M-K)^2 - 2K\log(1-u)} - (M-K)}{2M}. \tag{3.8}$$

Thus, $q(M,K,u)$ is the approximate uth quantile of $\tilde{W}$. The exact quartiles (corresponding to $u = \frac{1}{4}$, $\frac{2}{4}$ and $\frac{3}{4}$) are compared to $q(M,K,u)$ for $M = 400$ and a range of K in Fig. 1. The approximation appears to be good throughout the range of K.

Sengupta [17] appears to be the first to have discovered the Brownian bridge approximation appearing on the right of (3.7), and it was independently reported in [12] and [8]. The form of the error term given here is new.

4 ASYMPTOTIC DISTRIBUTION OF $W_{\max}$

Let both the empirical process U_K and the Brownian bridge B_0 be extended to processes on the real line $\mathcal{R}$ by making them periodic with period one. It is clear from the definition of U_K that it has stationary increments. The extended Brownian bridge also has stationary increments, as can be checked directly by examining the correlation function of this Gaussian process, but it also follows from the fact that B_0 is the weak limit of the empirical process as $K \to \infty$. Of course (3.3) continues to hold when the sup is taken over all $\mathcal{R}$, and this is again written symbollically as (3.4). Combining (1.3), (1.5) and (3.4) yields the following representation of $W_{\max}$:

$$W_{\max} = \max_{0 \leq s \leq t \leq 2} \left\{ \frac{\sqrt{K}}{M}(B_0(t) - B_0(s)) - \frac{M-K}{M}(t-s) \right\} + \frac{2O(\log K)}{M}. \tag{4.1}$$

Thus, the distribution of $\frac{M}{\sqrt{K}} W_{\max}$ is approximately the same as that of

$$M_\alpha = \max_{0 \leq s \leq t \leq 2} B_\alpha(t) - B_\alpha(s) \tag{4.2}$$

where B_α is defined by $B_\alpha(t) = B_0(t) - \alpha t$ for all t and $\alpha = \frac{M-K}{\sqrt{K}}$. In the remainder of this section we describe a method for computing the distribution of M_α. To begin with, note that in case $\alpha = 0$ (corresponding to $M = K$), the periodicity of B_0 yields that $M_0 = \max B_0 - \min B_0$. That is, M_0

is the range of the Brownian bridge, known to have the same distribution as the maximum of the Brownian excursion (see [20]), so that [10],

$$P[M_0 \leq c] = 1 - \sum_{n=1}^{\infty}(4n^2c^2 - 1)\exp(-2n^2c^2). \tag{4.3}$$

Assume for the remainder of the section that $\alpha > 0$. We present an alternate representation of the process B_α, and the representation of M_α as a corollary. Let B be a standard Brownian motion and let Θ be uniform on the interval $[0,1]$, and independent of B.

Proposition

$$[B(\cdot + \Theta \bmod 1) - B(\Theta) - \alpha\lfloor\cdot\rfloor \mid \min\{t \geq 0 : B(t) = -\alpha\} = 1] =_d B_\alpha \tag{4.4}$$

Proof. Let $(S_j)_{j=0}^{2n}$ be a symmetric Bernoulli random walk starting at the origin. Extend $(S_j)_{j=0}^{2n}$ to a doubly infinite sequence $(S_j : -\infty < j < \infty)$ by requiring $S_{j+2n} = S_j + S_{2n}$ for all j. Finally, let $(S(t) : t \in \mathcal{R})$ be defined by $S(j) = S_j$ for integers j and by linear interpolation between integer ordinates. Then in $C(\mathcal{R})$ as n tends to infinity

$$[(2n)^{-\frac{1}{2}}S(2n\cdot)|S_{2n} = -2\lceil\alpha\sqrt{\frac{n}{2}}\rceil] \to_d \ B_\alpha. \tag{4.5}$$

Define for $k \in \{1, 2, \ldots, 2n\}$ the event E_k by $E_k = \{S_k < S_j \text{ for } -\infty < j < k\}$. In other words, E_k is the event that the sequence $(S_j : -\infty < j < \infty)$ reaches a new minimum at index k. Let l be an integer with $1 \leq l \leq n$. Given that $S_{2n} = -2l$, exactly $2l$ of the events $E_1, \ldots, E_{2n}$ occur and each event has the same probability l/n. ¿From these facts it follows that the distribution of $[S|S_{2n} = -2l]$ is a convex combination of $2n$ distributions, namely the distributions of $[S|S_{2n} = -2l, E_k]$. Let Θ_{2n} be uniform on $\{0, \frac{1}{2n}, \cdots, \frac{2n-1}{2n}\}$. Since S has stationary increments the process

$$[(2n)^{-\frac{1}{2}}(S(2n(\cdot + \Theta_{2n})) - S(2n\Theta_{2n}))|S_{2n} = -2l, E_k]$$

has the same distribution for all k, and hence for all k it has the same distribution as $[(2n)^{-\frac{1}{2}}S|S_{2n} = -2l]$. In particular, $\{S_{2n} = -2l\} \cap E_{2n} = \{\min\{k \geq 0 : S_k = -2l\} = 2n\}$ so that

$$[(2n)^{-\frac{1}{2}}(S(2n(\cdot+\Theta_{2n}))-S(2n\Theta_{2n}))|\min\{k \geq 0 : S_k = -2l\} = 2n\}] =_d [(2n)^{-\frac{1}{2}}S|S_{2n} = -2l] \tag{4.6}$$

Take $2l = 2\lceil \alpha\sqrt{\frac{n}{2}}\rceil$ and let $n \to +\infty$. Use the facts [9]

$$(\Theta_{2n}, [(2n)^{-\frac{1}{2}}S|\min\{k : S_k = -2l\} = 2n]) \to_d (\Theta, [B - \alpha\lfloor\cdot\rfloor|\min\{t \geq 0|B(t) = -\alpha\} = 1])$$

and

$$[(2n)^{-\frac{1}{2}}S|S_{2n} = -2l] \to_d B_\alpha$$

to obtain (4.4) from (4.6) as $n \to +\infty$. $\Box$

Remark. By taking $l \equiv 1$ rather than $l = \lceil \alpha\sqrt{\frac{n}{2}}\rceil$ in the above proof, one readily establishes that

$$B_0^{\oplus}(\cdot + \Theta \text{ mod } 1) - B_0^{\oplus}(\Theta) =_d B_0$$

where $B_0^{\oplus}$ is a Brownian excursion. Since the Brownian excursion is strictly positive over (0,1), it follows immediately that the Brownian bridge $(B_0(t) : 0 \leq t \leq 1)$ reaches its minimum value at a unique time τ_m with probability 1, given by $\tau_m = 1 - \Theta$. Consequently,

$$B_0^{\oplus} =_d B_0(\tau_m + \cdot \text{ mod } 1) - B_0(\tau_m)$$

as shown by Vervaat [20] with a longer proof.

Let $(\tau_t : t \geq 0)$ denote the local time at zero of the standard Brownian motion B, and let τ^{-1} denote its inverse. Note that $\{\tau^{-1}(\alpha) = 1\}$ is equivalent to the event $\min\{s : \tau(s) = 1\} = \alpha$.

Corollary. $M_\alpha =_d [\ \max_{0\leq t\leq 1}|B(t)|\ \ |\tau^{-1}(\alpha) = 1]$

Proof. The value of M_α given in the definition (4.2) depends only on the increments of B_α and is invariant with respect to time shifts of B_α. Hence, the Proposition yields that

$$M_\alpha =_d [\ \max_{s\leq t} B(t \text{ mod } 1) - B(s \text{ mod } 1) - \alpha(\lfloor t\rfloor - \lfloor s\rfloor)\ |\ \min\{t \geq 0 : B(t) = -\alpha\} = 1]$$

The maximization can be restricted to be taken over $0 \leq s \leq t \leq 1$ with no effect, which implies that

$$\begin{aligned} M_\alpha &=_d [\max_{0\leq s\leq t} B(t) - B(s)|\min\{t \geq 0 : B(t) = -\alpha\} = 1] \\ &=_d [\max_{0\leq t\leq 1} B(t) - \underline{B}(t)|\min\{t \geq 0 : \underline{B}(t) = -\alpha\} = 1] \end{aligned}$$

where $\underline{B}(t) = \min_{0 \le s \le t} B(s)$. By Applying Levy's equivalence [10], $(B - \underline{B}, -\underline{B}) =_d (|B|, \tau)$, the corollary follows. $\Box$

The Corollary suggests that the distribution of M_α can be calculated by considering the Poisson process of excursions of Brownian motion indexed by inverse local time. The method bears similarity to that given in Section 2 for computing the distribution of $W_{\max}$ for K and M finite. Specifically, letting

$$f_{\alpha,\beta}(x)dx = P[\tau^{-1}(\alpha) \in [x, x+dx), |B(t)| \le \beta \text{ for } 0 \le t \le x]. \tag{4.7}$$

we have

$$P[M_\alpha \le \beta] = \frac{f_{\alpha,\beta}(1)}{f_{\alpha,\infty}(1)}. \tag{4.8}$$

The density function appearing in the denominator of (4.8) is simply the density of $\tau^{-1}(\alpha)$, or by Levy's equivalence the density of the hitting time of level $-\alpha$ by B, given by

$$f_{\alpha,\infty}(x) = \frac{\alpha}{\sqrt{2\pi}} x^{-\frac{3}{2}} \exp(-\frac{\alpha^2}{2x}). \tag{4.9}$$

The density $f_{\alpha,\beta}$ appearing in the numerator of (4.8) is studied in [11]. It is shown that the density has Laplace transform $\exp(-\alpha\sqrt{2s}\mathrm{cotanh}(\beta\sqrt{2s}))$. The required value $f_{\alpha,\beta}(1)$ can perhaps be computed by numerical inversion of this transform. Knight gives an analytical inversion of the transform as well, which we describe next. First, $\exp(\frac{\alpha}{\beta})f_{\alpha,\beta}$ is a probability density function, since $P[B(t) \le \beta, 0 \le t \le \tau^{-1}(\alpha)] = \exp(-\frac{\alpha}{\beta})$. Furthermore, Knight [11] showed that a random variable with this density can be represented by

$$\frac{2\beta^2}{\pi^2} \sum_{n=1}^{\infty} \frac{X_{n1} + \ldots + X_{nY_n}}{n^2}, \tag{4.10}$$

where $Y_n, X_{n,i}$, $n \ge 1, i \ge 1$ are mutually independent random variable, Y_n has the Poisson distribution with mean $\frac{2\alpha}{\beta}$ and $X_{n,i}$ is an exponential random variable with mean 1 for each i. Numerical computation of the distribution of M_α based on this representation, (4.8), and (4.9) is feasible but a little tricky, so some remarks are given about it here. Computation of the density $f_{\alpha,\beta}$ based on the representation (4.10) requires the convolution of infinitely many densities. However, as n increases the densities correspond to variables that quickly converge stochastically to zero. Thus a good approximation with controllable error is to replace the terms in the sum in (4.10) by their means for all large n. The effect of the tail terms is then incorporated by slightly shifting to the right the density computed after a finite number of convolutions. When the distributions are convolved numerically, they must be approximated by discrete probability distributions on a grid. Care must be taken to make sure the approximating distributions are actually probability distributions. One numerical method is to simply rescale the approximating distributions. A method we found to be more accurate is to approximate the exponential distributions by scaled geometric distributions.

Figure 2 shows the distribution functions of M_α for α=2,1 and 0, respectively from left to right. The curve for $\alpha = 0$ is given by (4.3), and the curves for $\alpha = 1$ and $\alpha = 2$ were obtained by computing convolutions over a grid of 10^5 points in the interval $[0, 1]$. Also shown are the distribution functions of the maximum work in a system with Bernoulli arrivals (in K slots two units of work arrive, in $M - K$ slots no work arrives, and one unit of work is completed in each slot as long as work is available).

ACKNOWLEDGEMENT The author is grateful to Prof. Peter Whittle for general professional inspiration, and for leading him jogging down Roman roads. Dr. Upamanyu Madhow, in a paper presented in a graduate topics course conducted by the author in the Spring of 1990, exploited the connection between queueing and the empirical process to obtain the asymptotic distributions in case $K = M$. Prof. Frank Knight provided much guidance regarding the literature on functionals of Brownian motion. This work was supported by the National Science Foundation under Grant NCR 90 04355, and the Joint Services Electrionics Program under Contract N00014-90-J-1270.

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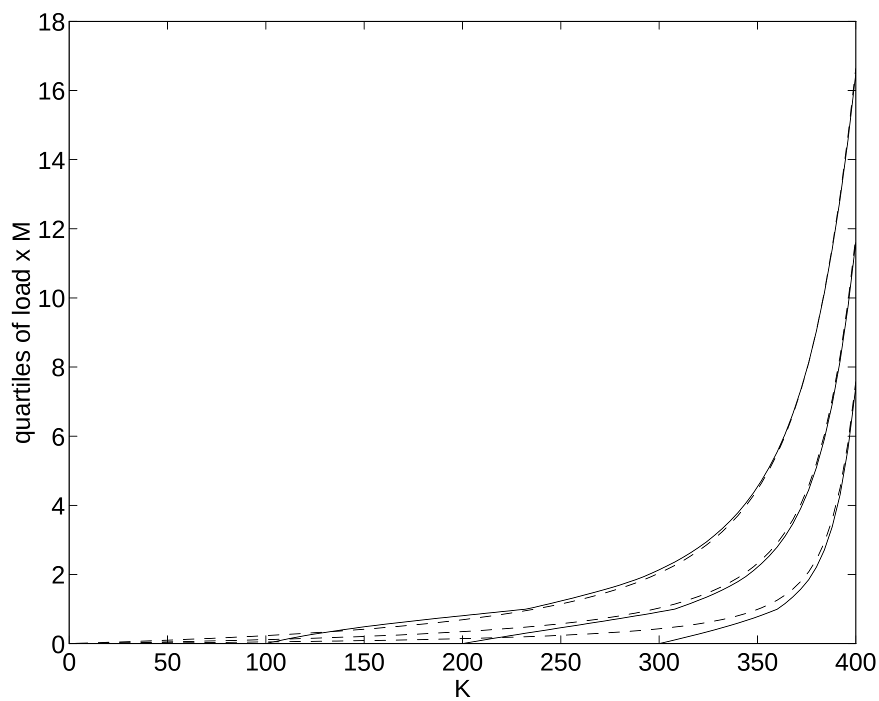

Figure 1: Actual quartiles of distribution of $M \times \tilde{W}$ (solid lines) and approximate quartiles (dashed lines) for $0 \leq K \leq M = 400$.

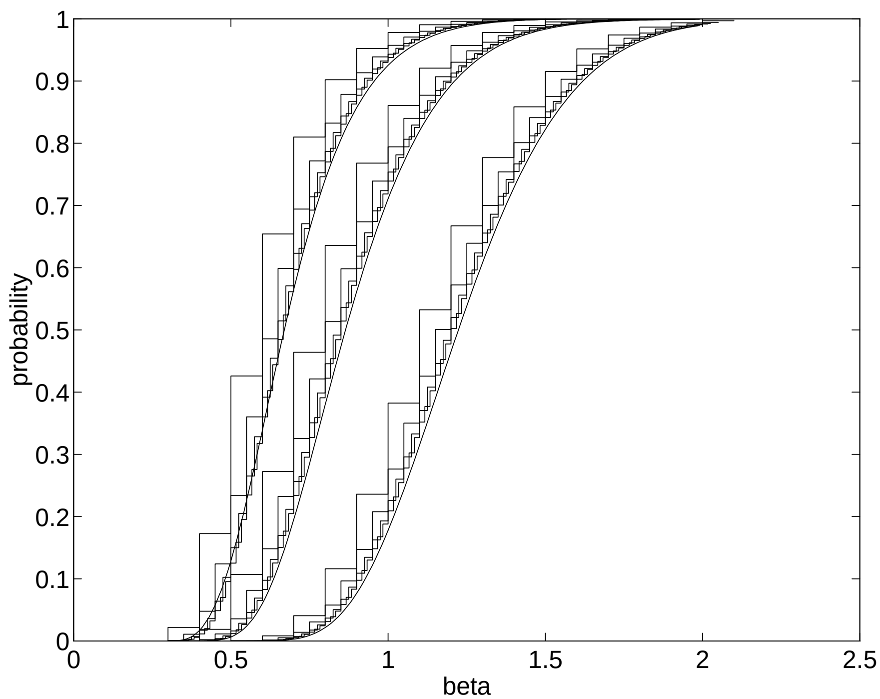

Figure 2: Distribution function of the maximum queue size normalized by division by $\sqrt{M}$. The three groups of curves correspond to $\alpha = 2, 1$ and 0, from left to right. The distribution for a Bernoulli model with M=100,400,1600 and 3600, and the limiting distribution, are shown for each value of α.