

Notes for ECE 567
Communication Network Analysis

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Preface

This is the latest draft of notes I have used for the graduate course *Communication Network Analysis*, offered by the Department of Electrical and Computer Engineering at the University of Illinois at Urbana-Champaign. The notes describe many of the most popular analytical techniques for design and analysis of computer communication networks, with an emphasis on performance issues such as delay, blocking, and resource allocation. Topics that are *not* covered in the notes include the Internet protocols (at least not explicitly), simulation techniques and simulation packages, and some of the mathematical proofs. These are covered in other books and courses.

The topics of these notes form a basis for understanding the literature on performance issues in networks, including the Internet. Specific topics include

- The basic and intermediate theory of queueing systems, along with stability criteria based on drift analysis and fluid models
- The notion of effective bandwidth, in which a constant bit rate equivalent is given for a bursty data stream in a given context
- An introduction to the calculus of deterministic constraints on traffic flows
- The use of penalty and barrier functions in optimization, and the natural extension to the use of utility functions and prices in the formulation of dynamic routing and congestion control problems
- Some topics related to performance analysis in wireless networks, including coverage of basic multiple access techniques, and transmission scheduling
- The basics of dynamic programming, introduced in the context of a simple queueing control problem
- The analysis of blocking and the reduced load fixed point approximation for circuit switched networks.

Students are assumed to have already had a course on computer communication networks, although the material in such a course is more to provide motivation for the material in these notes, than to provide understanding of the mathematics. In addition, since probability is used extensively, students in the class are assumed to have previously had two courses in probability. Some prior exposure to the theory of Lagrange multipliers for constrained optimization and nonlinear optimization algorithms is desirable, but not necessary.

I'm grateful to students and colleagues for suggestions and corrections, and am always eager for more.

Bruce Hajek, December 2006

Chapter 1

Countable State Markov Processes

1.1 Example of a Markov model

Consider a two-stage pipeline as pictured in Figure 1.1. Some assumptions about it will be made in order to model it as a simple discrete time Markov process, without any pretension of modeling a particular real life system. Each stage has a single buffer. Normalize time so that in one unit of time a packet can make a single transition. Call the time interval between k and $k + 1$ the k th “time slot,” and assume that the pipeline evolves in the following way during a given slot.

If at the beginning of the slot, there are no packets in stage one, then a new packet arrives to stage one with probability a , independently of the past history of the pipeline and of the outcome at state two.

If at the beginning of the slot, there is a packet in stage one and no packet in stage two, then the packet is transferred to stage two with probability d_1 .

If at the beginning of the slot, there is a packet in stage two, then the packet departs from the stage and leaves the system with probability d_2 , independently of the state or outcome of stage one.

These assumptions lead us to model the pipeline as a discrete-time Markov process with the state space $\mathcal{S} = \{00, 01, 10, 11\}$, transition probability diagram shown in Figure 1.2 (using the notation $\bar{x} = 1 - x$) and one-step transition probability matrix P given by

$$P = \begin{pmatrix} \bar{a} & 0 & a & 0 \\ \bar{a}d_2 & \bar{a}\bar{d}_2 & ad_2 & a\bar{d}_2 \\ 0 & d_1 & \bar{d}_1 & 0 \\ 0 & 0 & d_2 & \bar{d}_2 \end{pmatrix}$$

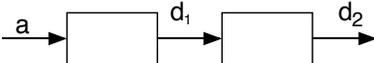


Figure 1.1: A two-stage pipeline

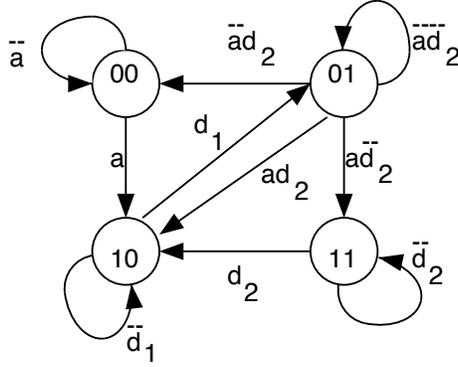


Figure 1.2: One-step transition probability diagram for example.

The rows of P are probability vectors. (In these notes, probability vectors are always taken to be row vectors, and more often than not, they are referred to as probability distributions.). For example, the first row is the probability distribution of the state at the end of a slot, given that the state is 00 at the beginning of a slot. Now that the model is specified, let us determine the throughput rate of the pipeline.

The equilibrium probability distribution $\pi = (\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11})$ is the probability vector satisfying the linear equation $\pi = \pi P$. Once π is found, the throughput rate η can be computed as follows. It is defined to be the rate (averaged over a long time) that packets transit the pipeline. Since at most two packets can be in the pipeline at a time, the following three quantities are all clearly the same, and can be taken to be the throughput rate.

The rate of arrivals to stage one

The rate of departures from stage one (or rate of arrivals to stage two)

The rate of departures from stage two

Focus on the first of these three quantities. Equating long term averages with statistical averages yields

$$\begin{aligned} \eta &= P[\text{an arrival at stage 1}] \\ &= P[\text{an arrival at stage 1} | \text{stage 1 empty at slot beginning}] P[\text{stage 1 empty at slot beginning}] \\ &= a(\pi_{00} + \pi_{01}). \end{aligned}$$

Similarly, by focusing on departures from stage 1, obtain $\eta = d_1\pi_{10}$. Finally, by focusing on departures from stage 2, obtain $\eta = d_2(\pi_{01} + \pi_{11})$. These three expressions for η must agree.

Consider the numerical example $a = d_1 = d_2 = 0.5$. The equation $\pi = \pi P$ yields that π is proportional to the vector $(1, 2, 3, 1)$. Applying the fact that π is a probability distribution yields that $\pi = (1/7, 2/7, 3/7, 1/7)$. Therefore $\eta = 3/14 = 0.214\dots$

By way of comparison, consider another system with only a single stage, containing a single buffer. In each slot, if the buffer is empty at the beginning of a slot an arrival occurs with probability a , and if the buffer has a packet at the beginning of a slot it departs with probability d . Simultaneous arrival and departure is not allowed. Then $\mathcal{S} = \{0, 1\}$, $\pi = (d/(a+d), a/(a+d))$ and the throughput

rate is $ad/(a+d)$. The two-stage pipeline with $d_2 = 1$ is essentially the same as the one-stage system. In case $a = d = 0.5$, the throughput rate of the single stage system is 0.25, which as expected is somewhat greater than that of the two-stage pipeline.

1.2 Definition, Notation and Properties

Having given an example of a discrete state Markov process, we now digress and give the formal definitions and some of the properties of Markov processes. Let \mathbb{T} be a subset of the real numbers \mathbb{R} and let \mathcal{S} be a finite or countably infinite set. A collection of \mathcal{S} -valued random variables $(X(t) : t \in \mathbb{T})$ is a *discrete-state Markov process* with state space \mathcal{S} if

$$P[X(t_{n+1}) = i_{n+1} | X(t_n) = i_n, \dots, X(t_1) = i_1] = P[X(t_{n+1}) = i_{n+1} | X(t_n) = i_n] \quad (1.1)$$

whenever

$$\begin{cases} t_1 < t_2 < \dots < t_{n+1} \text{ are in } \mathbb{T}, \\ i_1, i_2, \dots, i_{n+1} \text{ are in } \mathcal{S}, \text{ and} \\ P[X(t_n) = i_n, \dots, X(t_1) = i_1] > 0. \end{cases} \quad (1.2)$$

Set $p_{ij}(s, t) = P[X(t) = j | X(s) = i]$ and $\pi_i(t) = P[X(t) = i]$. The probability distribution $\pi(t) = (\pi_i(t) : i \in \mathcal{S})$ should be thought of as a row vector, and can be written as one once \mathcal{S} is ordered. Similarly, $H(s, t)$ defined by $H(s, t) = (p_{ij}(s, t) : i, j \in \mathcal{S})$ should be thought of as a matrix. Let e denote the column vector with all ones, indexed by \mathcal{S} . Since $\pi(t)$ and the rows of $H(s, t)$ are probability vectors for $s, t \in \mathbb{T}$ and $s \leq t$, it follows that $\pi(t)e = 1$ and, $H(s, t)e = e$.

Next observe that the marginal distributions $\pi(t)$ and the transition probabilities $p_{ij}(s, t)$ determine all the finite dimensional distributions of the Markov process. Indeed, given

$$\begin{cases} t_1 < t_2 < \dots < t_n \text{ in } \mathbb{T}, \\ i_1, i_2, \dots, i_n \in \mathcal{S} \end{cases} \quad (1.3)$$

one writes

$$\begin{aligned} P[X(t_1) = i_1, \dots, X(t_n) = i_n] &= \\ &P[X(t_1) = i_1, \dots, X(t_{n-1}) = i_{n-1}]P[X(t_n) = i_n | X(t_1) = i_1, \dots, X(t_{n-1}) = i_{n-1}] \\ &= P[X(t_1) = i_1, \dots, X(t_{n-1}) = i_{n-1}]p_{i_{n-1}i_n}(t_{n-1}, t_n) \end{aligned}$$

Application of this operation $n - 2$ more times yields that

$$P[X(t_1) = i_1, X(t_2) = i_2, \dots, X(t_n) = i_n] = \pi_{i_1}(t_1)p_{i_1i_2}(t_1, t_2) \dots p_{i_{n-1}i_n}(t_{n-1}, t_n), \quad (1.4)$$

which shows that the finite dimensional distributions of X are indeed determined by $(\pi(t))$ and $(p_{ij}(s, t))$. From this and the definition of conditional probabilities it follows by straight substitution that

$$\begin{aligned} P[X(t_j) = i_j, \text{ for } 1 \leq j \leq n+l | X(t_n) = i_n] &= \\ P[X(t_j) = i_j, \text{ for } 1 \leq j \leq n | X(t_n) = i_n]P[X(t_j) = i_j, \text{ for } n \leq j \leq n+l | X(t_n) = i_n] \end{aligned} \quad (1.5)$$

whenever $P[X(t_n) = i_n] > 0$. Property (1.5) is equivalent to the Markov property. Note in addition that it has no preferred direction of time, simply stating that the past and future are conditionally

independent given the present. It follows that if X is a Markov process, the time reversal of X defined by $\tilde{X}(t) = X(-t)$ is also a Markov process.

A Markov process is *time homogeneous* if $p_{ij}(s, t)$ depends on s and t only through $t - s$. In that case we write $p_{ij}(t - s)$ instead of $p_{ij}(s, t)$, and $H_{ij}(t - s)$ instead of $H_{ij}(s, t)$.

Recall that a random process is *stationary* if its finite dimensional distributions are invariant with respect to translation in time. Referring to (1.4), we see that a time-homogeneous Markov process is stationary if and only if its one dimensional distributions $\pi(t)$ do not depend on t . If, in our example of a two-stage pipeline, it is assumed that the pipeline is empty at time zero and that $a \neq 0$, then the process is not stationary (since $\pi(0) = (1, 0, 0, 0) \neq \pi(1) = (1 - a, 0, a, 0)$), even though it is time homogeneous. On the other hand, a Markov random process that is stationary is time homogeneous.

Computing the distribution of $X(t)$ by conditioning on the value of $X(s)$ yields that $\pi_j(t) = \sum_i P[X(s) = i, X(t) = j] = \sum_i \pi_i(s)p_{ij}(s, t)$, which in matrix form yields that $\pi(t) = \pi(s)H(s, t)$ for $s, t \in \mathbb{T}$, $s \leq t$. Similarly, given $s < \tau < t$, computing the conditional distribution of $X(t)$ given $X(s)$ by conditioning on the value of $X(\tau)$ yields

$$H(s, t) = H(s, \tau)H(\tau, t) \quad s, \tau, t \in \mathbb{T}, \quad s < \tau < t. \quad (1.6)$$

The relations (1.6) are known as the Chapman-Kolmogorov equations.

If the Markov process is time-homogeneous, then $\pi(s + \tau) = \pi(s)H(\tau)$ for $s, s + \tau \in \mathbb{T}$ and $\tau \geq 0$. A probability distribution π is called an *equilibrium* (or invariant) distribution if $\pi H(\tau) = \pi$ for all $\tau \geq 0$.

Repeated application of the Chapman-Kolmogorov equations yields that $p_{ij}(s, t)$ can be expressed in terms of transition probabilities for s and t close together. For example, consider Markov processes with index set the integers. Then $H(n, k + 1) = H(n, k)P(k)$ for $n \leq k$, where $P(k) = H(k, k + 1)$ is the one-step transition probability matrix. Fixing n and using forward recursion starting with $H(n, n) = I$, $H(n, n + 1) = P(n)$, $H(n, n + 2) = P(n)P(n + 1)$, and so forth yields

$$H(n, l) = P(n)P(n + 1) \cdots P(l - 1)$$

In particular, if the process is time-homogeneous then $H(k) = P^k$ for all k for some matrix P , and $\pi(l) = P^{l-k}\pi(k)$ for $l \geq k$. In this case a probability distribution π is an equilibrium distribution if and only if $\pi P = \pi$.

In the next section, processes indexed by the real line are considered. Such a process can be described in terms of $p(s, t)$ with $t - s$ arbitrarily small. By saving only a linearization, the concept of generator matrix arises naturally.

1.3 Pure-Jump, Time-Homogeneous Markov Processes

Let \mathcal{S} be a finite or countably infinite set, and let $\Delta \notin \mathcal{S}$. A *pure-jump function* is a function $x : \mathbb{R}_+ \rightarrow \mathcal{S} \cup \{\Delta\}$ such that there is a sequence of times, $0 = \tau_0 < \tau_1 < \dots$, and a sequence of states, s_0, s_1, \dots with $s_i \in \mathcal{S}$, and $s_i \neq s_{i+1}$, $i \geq 0$, so that

$$x(t) = \begin{cases} s_i & \text{if } \tau_i \leq t < \tau_{i+1} \\ \Delta & \text{if } t \geq \tau^* \end{cases} \quad i \geq 0 \quad (1.7)$$

where $\tau^* = \lim_{i \rightarrow \infty} \tau_i$. If τ^* is finite it is said to be the explosion time of the function x . The example corresponding to $\mathcal{S} = \{0, 1, \dots\}$, $\tau_i = i/(i + 1)$ and $s_i = i$ is pictured in Fig. 1.3. Note

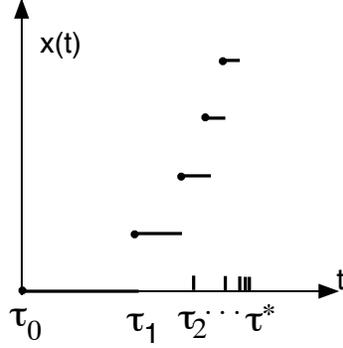


Figure 1.3: Sample pure-jump function with an explosion time

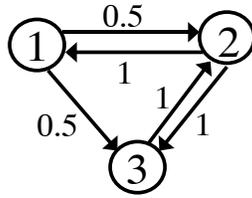


Figure 1.4: Transition rate diagram for a continuous time Markov process

that $\tau^* = 1$ for this example. A *pure-jump Markov process* $(X_t : t \geq 0)$ is a Markov process such that, with probability one, its sample paths are pure-jump functions.

Let $Q = (q_{ij} : i, j \in \mathcal{S})$ be such that

$$\begin{aligned} q_{ij} &\geq 0 & i, j \in \mathcal{S}, \quad i \neq j \\ q_{ii} &= -\sum_{j \in \mathcal{S}, j \neq i} q_{ij} & i \in \mathcal{S}. \end{aligned} \tag{1.8}$$

An example for state space $\mathcal{S} = \{1, 2, 3\}$ is

$$Q = \begin{pmatrix} -1 & 0.5 & 0.5 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

which can be represented by the transition rate diagram shown in Figure 1.4. A pure-jump, time-homogeneous Markov process X has *generator matrix* Q if

$$\lim_{h \searrow 0} (p_{ij}(h) - I_{\{i=j\}}) / h = q_{ij} \quad i, j \in \mathcal{S} \tag{1.9}$$

or equivalently

$$p_{ij}(h) = I_{\{i=j\}} + hq_{ij} + o(h) \quad i, j \in \mathcal{S} \tag{1.10}$$

where $o(h)$ represents a quantity such that $\lim_{h \rightarrow 0} o(h)/h = 0$.

For the example this means that the transition probability matrix for a time interval of duration h is given by

$$\begin{pmatrix} 1-h & 0.5h & 0.5h \\ h & 1-2h & h \\ 0 & h & 1-h \end{pmatrix} + \begin{pmatrix} o(h) & o(h) & o(h) \\ o(h) & o(h) & o(h) \\ o(h) & o(h) & o(h) \end{pmatrix}$$

The first term is a stochastic matrix, owing to the assumptions on the generator matrix Q .

Proposition 1.3.1 *Given a matrix Q satisfying (1.8), and a probability distribution $\pi(0) = (\pi_i(0) : i \in \mathcal{S})$, there is a pure-jump, time-homogeneous Markov process with generator matrix Q and initial distribution $\pi(0)$. The finite-dimensional distributions of the process are uniquely determined by $\pi(0)$ and Q .*

The proposition can be proved by appealing to the space-time properties in the next section. In some cases it can also be proved by considering the forward-differential evolution equations for $\pi(t)$, which are derived next. Fix $t > 0$ and let h be a small positive number. The Chapman-Kolmogorov equations imply that

$$\frac{\pi_j(t+h) - \pi_j(t)}{h} = \sum_{i \in \mathcal{S}} \pi_i(t) \left(\frac{p_{ij}(h) - I_{\{i=j\}}}{h} \right). \quad (1.11)$$

Consider letting h tend to zero. If the limit in (1.9) is uniform in i for j fixed, then the limit and summation on the right side of (1.11) can be interchanged to yield the forward-differential evolution equation:

$$\frac{\partial \pi_j(t)}{\partial t} = \sum_{i \in \mathcal{S}} \pi_i(t) q_{ij} \quad (1.12)$$

or $\frac{\partial \pi(t)}{\partial t} = \pi(t)Q$. This equation, known as the Kolmogorov forward equation, can be rewritten as

$$\frac{\partial \pi_j(t)}{\partial t} = \sum_{i \in \mathcal{S}, i \neq j} \pi_i(t) q_{ij} - \sum_{i \in \mathcal{S}, i \neq j} \pi_j(t) q_{ji}, \quad (1.13)$$

which states that the rate change of the probability of being at state j is the rate of “probability flow” into state j minus the rate of probability flow out of state j .

1.4 Space-Time Structure

Let $(X_k : k \in \mathbb{Z}_+)$ be a time-homogeneous Markov process with one-step transition probability matrix P . Let T_k denote the time that elapses between the k^{th} and $k+1^{\text{th}}$ jumps of X , and let $X^J(k)$ denote the state after k jumps. See Fig. 1.5 for illustration. More precisely, the *holding times* are defined by

$$T_0 = \min\{t \geq 0 : X(t) \neq X(0)\} \quad (1.14)$$

$$T_k = \min\{t \geq 0 : X(T_0 + \dots + T_{k-1} + t) \neq X(T_0 + \dots + T_{k-1})\} \quad (1.15)$$

and the *jump process* $X^J = (X^J(k) : k \geq 0)$ is defined by

$$X^J(0) = X(0) \quad \text{and} \quad X^J(k) = X(T_0 + \dots + T_{k-1}) \quad (1.16)$$

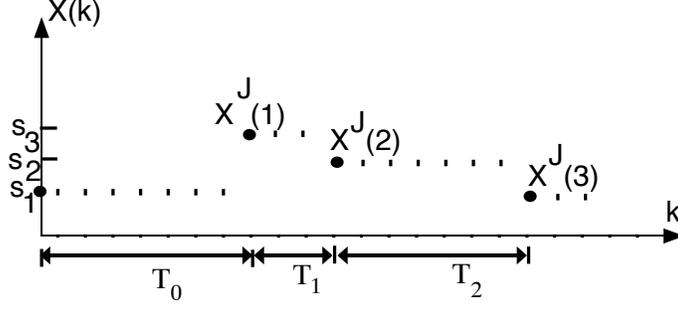


Figure 1.5: Illustration of jump process and holding times.

Clearly the holding times and jump process contain all the information needed to construct X , and vice versa. Thus, the following description of the joint distribution of the holding times and the jump process characterizes the distribution of X .

Proposition 1.4.1 *Let $X = (X(k) : k \in \mathbb{Z}_+)$ be a time-homogeneous Markov process with one-step transition probability matrix P .*

- (a) *The jump process X^J is itself a time-homogeneous Markov process, and its one-step transition probabilities are given by $p_{ij}^J = p_{ij}/(1 - p_{ii})$ for $i \neq j$, and $p_{ii}^J = 0$, $i, j \in \mathcal{S}$.*
- (b) *Given $X(0)$, $X^J(1)$ is conditionally independent of T_0 .*
- (c) *Given $(X^J(0), \dots, X^J(n)) = (j_0, \dots, j_n)$, the variables T_0, \dots, T_n are conditionally independent, and the conditional distribution of T_l is geometric with parameter $p_{j_l j_l}$:*

$$P[T_l = k | X^J(0) = j_0, \dots, X^J(n) = j_n] = p_{j_l j_l}^{k-1} (1 - p_{j_l j_l}) \quad 0 \leq l \leq n, k \geq 1.$$

Proof. Observe that if $X(0) = i$, then

$$\{T_0 = k, X^J(1) = j\} = \{X(1) = i, X(2) = i, \dots, X(k-1) = i, X(k) = j\},$$

so

$$P[T_0 = k, X^J(1) = j | X(0) = i] = p_{ii}^{k-1} p_{ij} = [(1 - p_{ii}) p_{ii}^{k-1}] p_{ij}^J \quad (1.17)$$

Because for i fixed the last expression in (1.17) displays the product of two probability distributions, conclude that *given* $X(0) = i$,

T_0 has distribution $((1 - p_{ii}) p_{ii}^{k-1} : k \geq 1)$, the geometric distribution of mean $1/(1 - p_{ii})$

$X^J(1)$ has distribution $(p_{ij}^J : j \in \mathcal{S})$ (i fixed)

T_0 and $X^J(1)$ are independent

More generally, check that

$$P[X^J(1) = j_1, \dots, X^J(n) = j_n, T_0 = k_0, \dots, T_n = k_n | X^J(0) = i] = p_{ij_1}^J p_{j_1 j_2}^J \cdots p_{j_{n-1} j_n}^J \prod_{l=0}^n (p_{j_l j_l}^{k_l-1} (1 - p_{j_l j_l}))$$

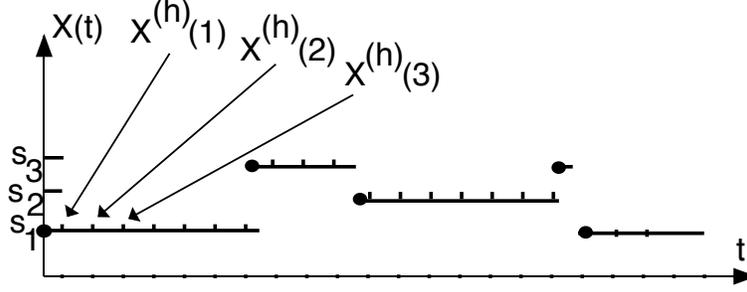


Figure 1.6: Illustration of sampling of a pure-jump function

This establishes the proposition. ■

Next we consider the space-time structure of time-homogeneous continuous-time pure-jump Markov processes. Essentially the only difference between the discrete- and continuous-time Markov processes is that the holding times for the continuous-time processes are exponentially distributed rather than geometrically distributed. Indeed, define the holding times $T_k, k \geq 0$ and the jump process X^J using (1.14)-(1.16) as before.

Proposition 1.4.2 *Let $X = (X(t) : t \in \mathbb{R}_+)$ be a time-homogeneous, pure-jump Markov process with generator matrix Q . Then*

- (a) *The jump process X^J is a discrete-time, time-homogeneous Markov process, and its one-step transition probabilities are given by*

$$p_{ij}^J = \begin{cases} -q_{ij}/q_{ii} & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases} \quad (1.18)$$

- (b) *Given $X(0)$, $X^J(1)$ is conditionally independent of T_0 .*

- (c) *Given $X^J(0) = j_0, \dots, X^J(n) = j_n$, the variables T_0, \dots, T_n are conditionally independent, and the conditional distribution of T_l is exponential with parameter $-q_{j_l j_l}$:*

$$P[T_l \geq c | X^J(0) = j_0, \dots, X^J(n) = j_n] = \exp(cq_{j_l j_l}) \quad 0 \leq l \leq n.$$

Proof. Fix $h > 0$ and define the “sampled” process $X^{(h)}$ by $X^{(h)}(k) = X(hk)$ for $k \geq 0$. See Fig. 1.6. Then $X^{(h)}$ is a discrete time Markov process with one-step transition probabilities $p_{ij}(h)$ (the transition probabilities for the original process for an interval of length h). Let $(T_k^{(h)} : k \geq 0)$ denote the sequence of holding times and $(X^{J,h}(k) : k \geq 0)$ the jump process for the process $X^{(h)}$.

The assumption that with probability one the sample paths of X are pure-jump functions, implies that *with probability one:*

$$\lim_{h \rightarrow 0} (X^{J,h}(0), X^{J,h}(1), \dots, X^{J,h}(n), hT_0^{(h)}, hT_1^{(h)}, \dots, hT_n^{(h)}) = (X^J(0), X^J(1), \dots, X^J(n), T_0, T_1, \dots, T_n) \quad (1.19)$$

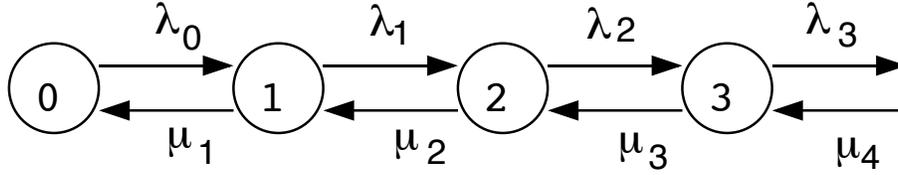


Figure 1.7: Transition rate diagram of a birth-death process

Since convergence with probability one implies convergence in distribution, the goal of identifying the distribution of the random vector on the righthand side of (1.19) can be accomplished by identifying the limit of the distribution of the vector on the left.

First, the limiting distribution of the process $X^{J,h}$ is identified. Since $X^{(h)}$ has one-step transition probabilities $p_{ij}(h)$, the formula for the jump process probabilities for discrete-time processes (see Proposition 1.4.1, part a) yields that the one step transition probabilities $p_{ij}^{J,h}$ for $X^{(J,h)}$ are given by

$$\begin{aligned} p_{ij}^{J,h} &= \frac{p_{ij}(h)}{1 - p_{ii}(h)} \\ &= \frac{p_{ij}(h)/h}{(1 - p_{ii}(h))/h} \rightarrow \frac{q_{ij}}{-q_{ii}} \text{ as } h \rightarrow 0 \end{aligned} \quad (1.20)$$

for $i \neq j$, where the limit indicated in (1.20) follows from the definition (1.9) of the generator matrix Q . Thus, the limiting distribution of $X^{J,h}$ is that of a Markov process with one-step transition probabilities given by (1.18), establishing part (a) of the proposition. The conditional independence properties stated in (b) and (c) of the proposition follow in the limit from the corresponding properties for the jump process $X^{J,h}$ guaranteed by Proposition 1.4.1. Finally, since $\log(1 + \theta) = \theta + o(\theta)$ by Taylor's formula, we have for all $c \geq 0$ that

$$\begin{aligned} P[hT_l^{(h)} > c | X^{J,h}(0) = j_0, \dots, X^{J,h} = j_n] &= (p_{j_l j_l}(h))^{\lfloor c/h \rfloor} \\ &= \exp(\lfloor c/h \rfloor \log(p_{j_l j_l}(h))) \\ &= \exp(\lfloor c/h \rfloor (q_{j_l j_l} h + o(h))) \\ &\rightarrow \exp(q_{j_l j_l} c) \text{ as } h \rightarrow 0 \end{aligned}$$

which establishes the remaining part of (c), and the proposition is proved. ■

Birth-Death Processes A useful class of countable state Markov processes is the set of birth-death processes. A (continuous time) birth-death process with parameters $(\lambda_0, \lambda_2, \dots)$ and (μ_1, μ_2, \dots) (also set $\lambda_{-1} = \mu_0 = 0$) is a pure-jump Markov process with state space $\mathcal{S} = \mathbb{Z}_+$ and generator matrix Q defined by $q_{kk+1} = \lambda_k$, $q_{kk} = -(\mu_k + \lambda_k)$, and $q_{kk-1} = \mu_k$ for $k \geq 0$, and $q_{ij} = 0$ if $|i - j| \geq 2$. The transition rate diagram is shown in Fig. (1.7). The space-time structure of such a process is as follows. Given the process is in state k at time t , the next state visited is $k + 1$ with probability $\lambda_k / (\lambda_k + \mu_k)$ and $k - 1$ with probability $\mu_k / (\lambda_k + \mu_k)$. The holding time of state k is exponential with parameter $\lambda_k + \mu_k$.

The space-time structure just described can be used to show that the limit in (1.9) is uniform in i for j fixed, so that the Kolmogorov forward equations are satisfied. These equations are:

$$\frac{\partial \pi_k(t)}{\partial t} = \lambda_{k-1} \pi_{k-1}(t) - (\lambda + \mu) \pi_k(t) + \mu_{k+1} \pi_{k+1}(t) \quad (1.21)$$

1.5 Poisson Processes

A *Poisson process with rate λ* is a birth-death process $N = (N(t) : t \geq 0)$ with initial distribution $P[N(0) = 0] = 1$, birth rates $\lambda_k = \lambda$ for all k and death rates $\mu_k = 0$ for all k . The space-time structure is particularly simple. The jump process N^J is deterministic and is given by $N^J(k) = k$. Therefore the holding times are not only conditionally independent given N^J , they are independent and each is exponentially distributed with parameter λ .

Let us calculate $\pi_j(t) = P[N(t) = j]$. The Kolmogorov forward equation for $k = 0$ is $\partial \pi_0 / \partial t = -\lambda \pi_0$, from which we deduce that $\pi_0(t) = \exp(-\lambda t)$. Next the equation for π_1 is $\partial \pi_1 / \partial t = \lambda \exp(-\lambda t) - \lambda \pi_1(t)$ which can be solved to yield $\pi_1(t) = (\lambda t) \exp(-\lambda t)$. Continuing by induction on k , verify that $\pi_k(t) = (\lambda t)^k \exp(-\lambda t) / k!$, so that $N(t)$ is a Poisson random variable with parameter k .

It is instructive to solve the Kolmogorov equations by another method, namely the z -transform method, since it works for some more complex Markov processes as well. For convenience, set $\pi_{-1}(t) = 0$ for all t . Then the Kolmogorov equations for π become $\frac{\partial \pi_k}{\partial t} = \lambda \pi_{k-1} - \lambda \pi_k$. Multiplying each side of this equation by z^k , summing over k , and interchanging the order of summation and differentiation, yields that the z transform $P^*(z, t)$ of $\pi(t)$ satisfies $\frac{\partial P^*(z, t)}{\partial t} = (\lambda z - \lambda) P^*(z, t)$. Solving this with the initial condition $P^*(z, 0) = 1$ yields that $P^*(z, t) = \exp((\lambda z - \lambda)t)$. Expanding $\exp(\lambda z t)$ into powers of z identifies $(\lambda t)^k \exp(-\lambda t) / k!$ as the coefficient of z^k in $P^*(z, t)$.

In general, for i fixed, $p_{ij}(t)$ is determined by the same equations but with the initial distribution $p_{ij}(0) = I_{\{i=j\}}$. The resulting solution is $p_{ij}(t) = \pi_{j-i}(t)$ for $j \geq i$ and $p_{ij}(t) = 0$ otherwise. Thus for $t_0 < t_1 < \dots < t_d = s < t$,

$$\begin{aligned} P[N(t) - N(s) = l | N(t_0) = i_0, \dots, N(t_{d-1}) = i_{d-1}, N(s) = i] \\ &= P[N(t) = i + l | N(s) = i] \\ &= [\lambda(t - s)]^l \exp(-\lambda(t - s)) / l! \end{aligned}$$

Conclude that $N(t) - N(s)$ is a Poisson random variable with mean $\lambda(t - s)$. Furthermore, $N(t) - N(s)$ is independent of $(N(u) : u \leq s)$, which implies that the increments $N(t_1) - N(t_0), N(t_2) - N(t_1), N(t_3) - N(t_2), \dots$ are mutually independent whenever $0 \leq t_0 < t_1 < \dots$.

Turning to another characterization, fix $T > 0$ and $\lambda > 0$, let U_1, U_2, \dots be uniformly distributed on the interval $[0, T]$, and let K be a Poisson random variable with mean λT . Finally, define the random process $(\tilde{N}(t) : 0 \leq t \leq T)$ by

$$\tilde{N}(t) = \sum_{i=1}^K I_{\{t \geq U_i\}}. \quad (1.22)$$

That is, for $0 \leq t \leq T$, $\tilde{N}(t)$ is the number of the first K uniform random variables located in $[0, t]$. We claim that \tilde{N} has the same distribution as a Poisson random process N with parameter λ , restricted to the interval $[0, T]$. To verify this claim, it suffices to check that the increments of the two

processes have the same joint distribution. To that end, fix $0 = t_0 < t_1 \dots < t_d = T$ and nonnegative integers k_1, \dots, k_d . Define the event $A = \{N(t_1) - N(t_0) = k_1, \dots, N(t_d) - N(t_{d-1}) = k_d\}$, and define the event \tilde{A} in the same way, but with N replaced by \tilde{N} . It is to be shown that $P[A] = P[\tilde{A}]$.

The independence of the increments of N and the Poisson distribution of each increment of N implies that

$$P[A] = \prod_{i=1}^d (\lambda(t_i - t_{i-1}))^{k_i} \exp(-\lambda(t_i - t_{i-1}))/k_i! \quad (1.23)$$

On the other hand, the event \tilde{A} is equivalent to the condition that $K = k$, where $k = k_1 + \dots + k_d$, and that furthermore the k random variables U_1, \dots, U_k are distributed appropriately across the d subintervals. Let $p_i = (t_i - t_{i-1})/T$ denote the relative length of the i th subinterval $[t_{i-1}, t_i]$. The probability that the *first* k_1 points U_1, \dots, U_{k_1} fall in the first interval $[t_0, t_1]$, the next k_2 points fall in the next interval, and so on, is $p_1^{k_1} p_2^{k_2} \dots p_d^{k_d}$. However, there are $\binom{k}{k_1 \dots k_d} = \frac{k!}{k_1! \dots k_d!}$ equally likely ways to assign the variables U_1, \dots, U_k to the intervals with k_i variables in the i th interval for all i , so

$$P[\tilde{A}] = P[K = k] \binom{k}{k_1 \dots k_d} p_1^{k_1} \dots p_d^{k_d}, \quad (1.24)$$

Some rearrangement yields that $P[A] = P[\tilde{A}]$, as was to be shown. Therefore, \tilde{N} and N restricted to the time interval $[0, T]$ have the same distribution.

The above discussion provides some useful characterizations of a Poisson process, which are summarized in the following proposition. Suppose that $X = (X(t) : t \geq 0)$ has pure-jump sample paths with probability one, suppose $P[X(0) = 0] = 1$, and suppose $\lambda > 0$.

Proposition 1.5.1 *The following four are equivalent:*

- (i) X is a Poisson process with rate λ .
- (ii) The jumps of X are all size one, and the holding times are independent, exponentially distributed with parameter λ .
- (iii) Whenever $0 \leq t_0 < t_1 \dots < t_n$, the increments $X(t_1) - X(t_0), \dots, X(t_n) - X(t_{n-1})$ are independent random variables, and whenever $0 \leq s < t$, the increment $X(t) - X(s)$ is a Poisson random variable with mean $\lambda(t - s)$.
- (iv) Whenever $0 \leq s < t$, the increment $X(t) - X(s)$ is a Poisson random variable with mean $\lambda(t - s)$, and given $X(t) - X(s) = n$, the jumps of X in $[s, t]$ are height one and the vector of their locations is uniformly distributed over the set $\{\tau \in \mathbb{R}_+^n : s < \tau_1 < \dots < \tau_n < t\}$.

The fourth characterization of 1.5.1 can be readily extended to define Poisson point processes on any set with a suitable positive measure giving the mean number of points within subsets. For example, a Poisson point process on \mathbb{R}^d with intensity λ is specified by the following two conditions. First, for any subset A of \mathbb{R}^d with finite volume, the distribution of the number of points in the set is Poisson with mean $\lambda \cdot \text{Volume}(A)$. Second, given that the number of points in such a set A is n , the locations of the points within A are distributed according to n mutually independent random variables, each uniformly distributed over A . As a sample calculation, take $d = 2$ and consider the distance R of the closest point of the Poisson point process to the origin. Since the

number of points falling within a circle of radius r has the Poisson distribution with mean πr^2 , the distribution function of R is given by

$$P[R \leq r] = 1 - P[\text{no points of the process in circle of radius } r] = 1 - \exp(-\lambda \pi r^2).$$

1.6 Renewal Theory

Renewal theory describes convergence to statistical equilibrium for systems involving sequences of intervals of random duration. It also helps describe what is meant by the state of a random system at a typical time, or as viewed by a typical customer. We begin by examining Poisson processes and an illustration of the dependence of an average on the choice of sampling distribution.

Consider a two-sided Poisson process $(N(t) : t \in R)$ with parameter λ , which by definition means that $N(0) = 0$, N is right-continuous, the increment $N(t+s) - N(t)$ is a Poisson random variable with mean λs for any $s > 0$ and t , and the increments of N are independent. Fix a time t . With probability one, t is strictly in between two jump times of N . The time between t and the first jump after time t is exponentially distributed, with mean $1/\lambda$. Similarly, the time between the last jump before time t and t is also exponentially distributed with mean $1/\lambda$. (In fact, these two times are independent.) Thus, the mean of the length of the interval between the jumps surrounding t is $2/\lambda$. In contrast, the Poisson process is built up from exponential random variables with mean $1/\lambda$. That the two means are distinct is known as the ‘‘Poisson paradox.’’

Perhaps an analogy can help explain the Poisson paradox. Observe that the average population of a country on earth is about $6 \times 10^9 / 200 = 30$ million people. On the other hand, if a person is chosen at random from among all people on earth, the mean number of people in the same country as that person is greater than 240 million people. Indeed, the population of China is about 1.2×10^9 , so that about one of five people live in China. Thus, the mean number of people in the same country as a randomly selected person is at least $1.2 \times 10^9 / 5 = 240$ million people. The second average is much larger than the first, because sampling a country by choosing a person at random skews the distribution of size towards larger countries in direct proportion to the populations of the countries. In the same way, the sampled lifetime of a Poisson process is greater than the typical lifetimes used to generate the process.

A generalization of the Poisson process is a counting process in which the times between arrivals are independent and identically distributed, but not necessarily exponentially distributed. For slightly more generality, the time until the first arrival is allowed to have a distribution different from the lengths of times between arrivals, as follows.

Suppose F and G are distribution functions of nonnegative random variables, and to avoid trivialities, suppose that $F(0) < 1$. Let T_1, T_2, \dots be independent, such that T_1 has distribution function G and T_k has distribution function F for $k \geq 2$. The time of the n^{th} arrival is $\tau(n) = T_1 + T_2 + \dots + T_n$ and the number of arrivals up to time t is $\alpha(t) = \max\{n : \tau(n) \leq t\}$. The *residual lifetime* at a fixed time t is defined to be $\gamma_t = \tau(\alpha(t) + 1) - t$ and the *sampled lifetime* at a fixed time t is $L_t = T_{\alpha(t)+1}$, as pictured in Figure 1.8. ¹ Renewal theory, the focus of this section, describes the limits of the residual lifetime and sampled lifetime distributions for such processes.

This section concludes with a presentation of the renewal equation, which plays a key role in renewal theory. To write the equation in general form, the definition of convolution of two measures on the real line, given next, is needed. Let F_1 and F_2 be two distribution functions, corresponding

¹Note that if one or more arrivals occur at a time t , then γ_t and L_t are both equal to the next nonzero lifetime.

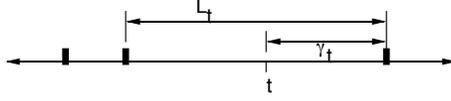


Figure 1.8: Sampled lifetime and forward lifetime

to two positive measures on the real line. The convolution of the two measures is again a measure, such that its distribution function is $F_1 * F_2$, defined by the Riemann-Stieltjes integral

$$F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x - y)F_2(dy)$$

If X_1 and X_2 are independent random variables, if X_1 has distribution function F_1 , and if X_2 has distribution function F_2 , then $X_1 + X_2$ has distribution function $F_1 * F_2$. It is clear from this that convolution is commutative ($F * G = G * F$) and associative ($(F * G) * H = F * (G * H)$). If F_1 has density function f_1 and F_2 has density function F_2 , then $F_1 * F_2$ has density function $f_1 * f_2$, defined as the usual convolution of functions on \mathbb{R} :

$$f_1 * f_2(x) = \int_{-\infty}^{\infty} f_1(x - y)f_2(y)dy$$

Similarly, if F_1 has a discrete mass function f_1 on \mathbb{Z} and F_2 has discrete mass function f_2 on \mathbb{Z} , then $F_1 * F_2$ has discrete mass function $f_1 * f_2$ defined as the usual convolution of functions on \mathbb{Z} :

$$f_1 * f_2(n) = \sum_{k=-\infty}^{\infty} f_1(n - k)f_2(k).$$

Define the cumulative mean renewal function H by $H(t) = E[\alpha(t)]$. Function H can be viewed as the cumulative distribution function for the renewal measure, which is the measure on \mathbb{R}_+ such that the measure of an interval is the mean number of arrivals in the interval. In particular, for $0 \leq s \leq t$, $H(t) - H(s)$ is the mean number of arrivals in the interval $(s, t]$. The function H determines much about the renewal variables—in particular the joint distribution of (L_t, γ_t) can be expressed in terms of H and the CDFs F and G . For example, if $t \geq u > 0$, then

$$\begin{aligned} P[L_t \leq u] &= \int_{t-u}^t P[\text{a renewal occurs in } [s, s + ds] \text{ with lifetime} \in [t - s, u]] \\ &= \int_{t-u}^t (F(u) - F(t - s))dH(s). \end{aligned} \tag{1.25}$$

Observe that the function H can be expressed as

$$H(t) = \sum_{n=1}^{\infty} U_n(t) \tag{1.26}$$

where U_n is the probability distribution function of $T_1 + \dots + T_n$. From (1.26) and the facts $U_1 = G$ and $U_n = F * U_{n-1}$ for $n \geq 2$, the *renewal equation* follows:

$$H = G + F * H$$

1.6.1 Renewal Theory in Continuous Time

The distribution function F is said to be *lattice type* if, for some $a > 0$, the distribution is concentrated on the set of integer multiples of a . Otherwise F is said to be *nonlattice type*. In this subsection F is assumed to be nonlattice type, while the next subsection touches on discrete-time renewal theory, addressing the situations when F is lattice type. If F and G have densities, then H has a density and the renewal equation can be written as $h = g + f * h$. Many convergence questions about renewal processes, and also about Markov processes, can be settled by the following theorem (for a proof, see [1, 18]).

Proposition 1.6.1 (*Blackwell's Renewal Theorem in continuous time*) *Suppose F is a nonlattice distribution. Then for fixed $h > 0$,*

$$\lim_{t \rightarrow \infty} (H(t+h) - H(t))/h = 1/m_1,$$

where m_k is the k^{th} moment of F .

The renewal theorem can be used to show that if the distribution F is nonlattice, then

$$\lim_{t \rightarrow \infty} P[L_t \leq u, \gamma_t \leq s] = F_{L\gamma}(u, s)$$

where

$$F_{L\gamma}(u, s) = \frac{1}{m_1} \int_0^{s \wedge u} (F(u) - F(y)) dy \quad u, s \in \mathbb{R}_+ \quad (1.27)$$

For example, to identify $\lim_{t \rightarrow \infty} P[L_t \leq u]$ one can start with (1.25). Roughly speaking, the renewal theorem tells us that in the limit, we can replace $dH(s)$ in (1.25) by $\frac{ds}{m_1}$. This can be justified by transforming the representation (1.25) by a change of variables (letting $x = t - s$ for t fixed) and an integration by parts, before applying the renewal theorem, as follows

$$\begin{aligned} P[L_t \leq u] &= - \int_0^u (F(u) - F(x)) dH(t-x) \\ &= \int_0^u (H(t) - H(t-x)) dF(x) \\ &\rightarrow \int_0^u \frac{x}{m_1} dF(x) = \frac{1}{m_1} \int_0^u (F(u) - F(y)) dy \end{aligned}$$

The following properties of the joint distribution of (L, γ) under $F_{L\gamma}$ are usually much easier to work with than the expression (1.27):

Moments: $E[L^k] = m_{k+1}/m_1$ and $E[\gamma^k] = m_{k+1}/((k+1)m_1)$,

Density of γ : γ has density $f_\gamma(s) = (1 - F(s))/m_1$,

Joint density: If F has density f , then (L, γ) has density

$$f_{L,\gamma}(u, s) = \begin{cases} f(u)/m_1 & \text{if } 0 \leq s \leq u \\ 0 & \text{else} \end{cases}$$

Equivalently, L has density $f_L(u) = uf(u)/m_1$ and given $L = u$, γ is uniformly distributed over the interval $[0, u]$.

Laplace transforms: $F_\gamma^*(s) = (1 - F^*(s))/(sm_1)$ and $F_L^*(s) = -(dF^*(s)/ds)/m_1$.

The fact that the stationary sampled lifetime density, $f_L(u)$, is proportional to $uf(u)$, makes sense according to the intuition gleaned from the countries example. The lifetimes here play the role of countries in that example, and sampling by a particular time (analogous to selecting a country by first selecting an individual) results in a factor u proportional to the size of the lifetime.

1.6.2 Renewal Theory in Discrete Time

Suppose that the variables T_1, T_2, \dots are integer valued. The process $\alpha(t)$ need only be considered for integer values of t . The renewal equation can be written as $h = g + f * h$. where $h(n)$ is the expected number of renewals that occur at time n , and f and g are the probability mass functions corresponding to F and G . The z -transform of h is readily found to be given by $H(z) = G(z)/(1 - F(z))$.

Recall that the greatest common divisor, GCD, of a set of positive integers is the largest integer that evenly divides all the integers in the set. For example, $\text{GCD}\{4, 6, 12\} = 2$.

Proposition 1.6.2 (*Blackwell's Renewal Theorem in discrete time*) Suppose $\text{GCD}\{k \geq 1 : f(k) > 0\} = 1$. Then $\lim_{n \rightarrow \infty} h(n) = 1/m_1$, where m_k is the k^{th} moment of F .

The proposition can be used to prove the following fact. If $\text{GCD}\{k \geq 1 : f(k) > 0\} = 1$, then (L_n, γ_n) converges in distribution as $n \rightarrow \infty$, and under the limiting distribution

$$P[L = l] = lf_l/m_1 \quad \text{and} \quad P[\gamma = i | L = l] = 1/l \quad 1 \leq i \leq l,$$

$$P[\gamma = i] = \frac{1 - F(i-1)}{m_1} \quad i \geq 1$$

and

$$E[L^k] = m_{k+1}/m_1 \quad \text{and} \quad E[\gamma] = ((m_2/2m_1)) + 0.5.$$

Note that if a renewal occurs at time t , then by definition γ_t is the time between t and the first renewal after time t . An alternative definition would take γ_t equal to zero in such a case. Under the alternative definition, $E[\gamma] = ((m_2/2m_1)) - 0.5$. In fact, this is the same as the mean of the *backwards* residual arrival time for the original definition. In contrast, in continuous time, both the forward and backward residual lifetimes have mean $m_2/2m_1$.

1.7 Classification and Convergence of Discrete State Markov Processes

1.7.1 Examples with finite state space

Recall that a probability distribution π on \mathcal{S} is an *equilibrium probability distribution* for a time-homogeneous Markov process X if $\pi = \pi H(t)$ for all t . We shall see in this section that under certain natural conditions, the existence of an equilibrium probability distribution is related to whether the distribution of $X(t)$ converges as $t \rightarrow \infty$. Existence of an equilibrium distribution is also connected to the mean time needed for X to return to its starting state. To motivate the conditions that will be imposed, we begin by considering four examples of finite state processes.

Then the relevant definitions are given for finite or countably-infinite state space, and propositions regarding convergence are presented. The key role of renewal theory is exposed.

Example 1.1 Consider the discrete-time Markov process with the one-step probability diagram shown in Figure 1.9. Note that the process can't escape from the set of states $\mathcal{S}_1 = \{a, b, c, d, e\}$, so that if the initial state $X(0)$ is in \mathcal{S}_1 with probability one, then the limiting distribution is supported by \mathcal{S}_1 . Similarly if the initial state $X(0)$ is in $\mathcal{S}_2 = \{f, g, h\}$ with probability one, then the limiting distribution is supported by \mathcal{S}_2 . Thus, the limiting distribution is not unique for this process. The natural way to deal with this problem is to decompose the original problem into two problems. That is, consider a Markov process on \mathcal{S}_1 , and then consider a Markov process on \mathcal{S}_2 .

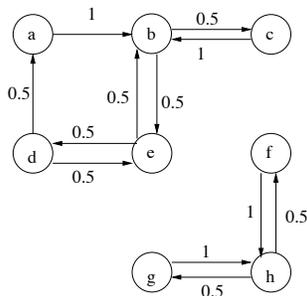


Figure 1.9: Example 1.1

Does the distribution of $X(k)$ necessarily converge if $X(0) \in \mathcal{S}_1$ with probability one? The answer is no. For example, note that if $X(0) = a$, then $X(k) \in \{a, c, e\}$ for all even values of k , whereas $X(k) \in \{b, d\}$ for all odd values of k . That is, $\pi_a(k) + \pi_c(k) + \pi_e(k)$ is one if k is even and is zero if k is odd. Therefore, if $\pi_a(0) = 1$, then $\pi(k)$ does not converge as $k \rightarrow \infty$.

Basically speaking, the Markov process of this example fails to have a unique limiting distribution independent of the initial state for two reasons: (i) the process is not irreducible, and (ii) the states are periodic.

Example 1.2 Consider the two-state, continuous time Markov process with the transition rate diagram shown in Figure 1.10 for some positive constants α and β . The generator matrix is given

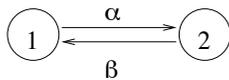


Figure 1.10: Example 1.2

by

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

We shall compute $H(t) = e^{Qt}$ by solving the Kolmogorov forward equations. The first equation is

$$\frac{\partial p_{11}(t)}{\partial t} = -\alpha p_{11}(t) + \beta p_{12}(t); \quad p_{11}(0) = 1$$

But $p_{12}(t) = 1 - p_{11}(t)$, so

$$\frac{\partial p_{11}(t)}{\partial t} = -(\alpha + \beta)p_{11}(t) + \beta; \quad p_{11}(0) = 1$$

By differentiation we check that this equation has the solution

$$\begin{aligned} p_{11}(t) &= e^{-(\alpha+\beta)t} + \int_0^t e^{-(\alpha+\beta)(t-s)} \beta ds \\ &= \frac{\beta}{\alpha + \beta} (1 - e^{-(\alpha+\beta)t}). \end{aligned}$$

Using the fact $p_{12}(t) = 1 - p_{11}(t)$ yields $p_{12}(t)$. The second row of $H(t)$ can be found similarly, yielding

$$H(t) = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} + e^{-(\alpha+\beta)t} \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & -\frac{\alpha}{\alpha+\beta} \\ -\frac{\beta}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix}$$

Note that $H(0) = I$ as required, and that

$$\lim_{t \rightarrow \infty} H(t) = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

Thus, for any initial distribution $\pi(0)$,

$$\lim_{t \rightarrow \infty} \pi(t) = \lim_{t \rightarrow \infty} \pi(0)H(t) = \left(\frac{\beta}{\alpha + \beta}, \frac{\alpha}{\alpha + \beta} \right).$$

The rate of convergence is exponential, with rate parameter $\alpha + \beta$, which happens to be the nonzero eigenvalue of Q . Note that the limiting distribution is the unique probability distribution satisfying $\pi Q = 0$. The periodicity problem of Example 1.1 does not arise for continuous-time processes.

Example 1.3 Consider the continuous-time Markov process with the transition rate diagram in Figure 1.11. The Q matrix is the block-diagonal matrix given by



Figure 1.11: Example 1.3

$$Q = \begin{bmatrix} -\alpha & \alpha & 0 & 0 \\ \beta & -\beta & 0 & 0 \\ 0 & 0 & -\alpha & \alpha \\ 0 & 0 & \beta & -\beta \end{bmatrix}$$

This process is not irreducible, but rather the transition rate diagram can be decomposed into two parts, each equivalent to the diagram for Example 1.1. The equilibrium probability distributions are the probability distributions of the form $\pi = (\lambda \frac{\beta}{\alpha+\beta}, \lambda \frac{\alpha}{\alpha+\beta}, (1-\lambda) \frac{\beta}{\alpha+\beta}, (1-\lambda) \frac{\alpha}{\alpha+\beta})$, where λ is the probability placed on the subset $\{1, 2\}$.

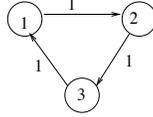


Figure 1.12: Example 1.4

Example 1.4 Consider the discrete-time Markov process with the transition probability diagram in Figure 1.12. The one-step transition probability matrix P is given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solving the equation $\pi = \pi P$ we find there is a unique equilibrium probability vector, namely $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. On the other hand, if $\pi(0) = (1, 0, 0)$, then

$$\pi(k) = \pi(0)P^k = \begin{cases} (1, 0, 0) & \text{if } k \equiv 0 \pmod{3} \\ (0, 1, 0) & \text{if } k \equiv 1 \pmod{3} \\ (0, 0, 1) & \text{if } k \equiv 2 \pmod{3} \end{cases}$$

Therefore, $\pi(k)$ does not converge as $k \rightarrow \infty$.

Let X be a discrete-time, or pure-jump continuous-time, Markov process on the countable state space \mathcal{S} . Suppose X is time-homogeneous. The process is said to be *irreducible* if for all $i, j \in \mathcal{S}$, there exists $s > 0$ so that $p_{ij}(s) > 0$.

1.7.2 Discrete Time

Clearly a probability distribution π is an equilibrium probability distribution for a discrete-time time-homogeneous Markov process X if and only if $\pi = \pi P$. The *period* of a state i is defined to be $\text{GCD}\{k \geq 0 : p_{ii}(k) > 0\}$. The set $\{k \geq 0 : p_{ii}(k) > 0\}$ is closed under addition, which by elementary algebra² implies that the set contains all sufficiently large integer multiples of the period. The Markov process is called *aperiodic* if the period of all the states is one.

Proposition 1.7.1 *If X is irreducible, all states have the same period.*

Proof. Let i and j be two states. By irreducibility, there are integers k_1 and k_2 so that $p_{ij}(k_1) > 0$ and $p_{ji}(k_2) > 0$. For any integer n , $p_{ii}(n + k_1 + k_2) \geq p_{ij}(k_1)p_{jj}(n)p_{ji}(k_2)$, so the set $\{k \geq 0 : p_{ii}(k) > 0\}$ contains the set $\{k \geq 0 : p_{jj}(k) > 0\}$ translated up by $k_1 + k_2$. Thus the period of i is less than or equal to the period of j . Since i and j were arbitrary states, the proposition follows. ■

For a fixed state i , define $\tau_i = \min\{k \geq 1 : X(k) = i\}$, where we adopt the convention that the minimum of an empty set of numbers is $+\infty$. Let $M_i = E[\tau_i | X(0) = i]$. If $P[\tau_i < +\infty | X(0) = i] < 1$, then state i is called *transient* (and by convention, $M_i = +\infty$). Otherwise $P[\tau_i < +\infty | X(0) = i] = 1$, and i is then said to be *positive - recurrent* if $M_i < +\infty$ and to be *null - recurrent* if $M_i = +\infty$.

²see Euclidean algorithm, Chinese remainder theorem, Bezout theorem

Proposition 1.7.2 *Suppose X is irreducible and aperiodic.*

- (a) *All states are transient, or all are positive-recurrent, or all are null-recurrent.*
- (b) *For any initial distribution $\pi(0)$, $\lim_{t \rightarrow \infty} \pi_i(t) = 1/M_i$, with the understanding that the limit is zero if $M_i = +\infty$.*
- (c) *An equilibrium probability distribution π exists if and only if all states are positive-recurrent.*
- (d) *If it exists, the equilibrium probability distribution π is given by $\pi_i = 1/M_i$. (In particular, if it exists, the equilibrium probability distribution is unique).*

Proof. (a) Suppose state i is recurrent. Given $X(0) = i$, after leaving i the process returns to state i at time τ_i . The process during the time interval $\{0, \dots, \tau_i\}$ is the first excursion of X from state i . From time τ_i onward, the process behaves just as it did initially. Thus there is a second excursion from i , third excursion from i , and so on. Let T_k for $k \geq 1$ denote the length of the k th excursion. Then the T_k 's are independent, and each has the same distribution as $T_1 = \tau_i$. Let j be another state and let ϵ denote the probability that X visits state j during one excursion from i . Since X is irreducible, $\epsilon > 0$. The excursions are independent, so state j is visited during the k th excursion with probability ϵ , independently of whether j was visited in earlier excursions. Thus, the number of excursions needed until state j is reached has the geometric distribution with parameter ϵ , which has mean $1/\epsilon$. In particular, state j is eventually visited with probability one. After j is visited the process eventually returns to state i , and then within an average of $1/\epsilon$ additional excursions, it will return to state j again. Thus, state j is also recurrent. Hence, if one state is recurrent, all states are recurrent.

The same argument shows that if i is positive-recurrent, then j is positive-recurrent. Given $X(0) = i$, the mean time needed for the process to visit j and then return to i is M_i/ϵ , since on average $1/\epsilon$ excursions of mean length M_i are needed. Thus, the mean time to hit j starting from i , and the mean time to hit i starting from j , are both finite. Thus, j is positive-recurrent. Hence, if one state is positive-recurrent, all states are positive-recurrent.

(b) Part (b) of the proposition follows immediately from the renewal theorem for discrete time. The first lifetime for the renewal process is the time needed to first reach state i , and the subsequent renewal times are the lengths of the excursions from state i . The probability $\pi_i(t)$ is the probability that there is a renewal event at time t , which by the renewal theorem converges to $1/M_i$.

(c) Suppose all states are positive-recurrent. By the law of large numbers, for any state j , the long run fraction of time the process is in state j is $1/M_j$ with probability one. Similarly, for any states i and j , the long run fraction of time the process is in state j is γ_{ij}/M_i , where γ_{ij} is the mean number of visits to j in an excursion from i . Therefore $1/M_j = \gamma_{ij}/M_i$. This implies that $\sum_i 1/M_i = 1$. That is, π defined by $\pi_i = 1/M_i$ is a probability distribution. The convergence for each i separately given in part (b), together with the fact that π is a probability distribution, imply that $\sum_i |\pi_i(t) - \pi_i| \rightarrow 0$. Thus, taking s to infinity in the equation $\pi(s)H(t) = \pi(s+t)$ yields $\pi H(t) = \pi$, so that π is an equilibrium probability distribution.

Conversely, if there is an equilibrium probability distribution π , consider running the process with initial state π . Then $\pi(t) = \pi$ for all t . So by part (b), for any state i , $\pi_i = 1/M_i$. Taking a state i such that $\pi_i > 0$, it follows that $M_i < \infty$. So state i is positive-recurrent. By part (a), all states are positive-recurrent.

(d) Part (d) was proved in the course of proving part (c). ■

We conclude this section by describing a technique to establish a rate of convergence to the equilibrium distribution for finite-state Markov processes. Define $\delta(P)$ for a one-step transition probability matrix P by

$$\delta(P) = \min_{i,k} \sum_j p_{ij} \wedge p_{kj},$$

where $a \wedge b = \min\{a, b\}$. The number $\delta(P)$ is known as Dobrushin's coefficient of ergodicity. Since $a + b - 2(a \wedge b) = |a - b|$ for $a, b \geq 0$, we also have

$$1 - 2\delta(P) = \min_{i,k} \sum_j |p_{ij} - p_{kj}|.$$

Let $\|\mu\|$ for a vector μ denote the L_1 norm: $\|\mu\| = \sum_i |\mu_i|$.

Proposition 1.7.3 *For any probability vectors π and σ , $\|\pi P - \sigma P\| \leq (1 - \delta(P))\|\pi - \sigma\|$. Furthermore, if $\delta(P) > 0$ then there is a unique equilibrium distribution π^∞ , and for any other probability distribution π on \mathcal{S} , $\|\pi P^l - \pi^\infty\| \leq 2(1 - \delta(P))^l$.*

Proof. Let $\tilde{\pi}_i = \pi_i - \pi_i \wedge \sigma_i$ and $\tilde{\sigma}_i = \sigma_i - \pi_i \wedge \sigma_i$. Clearly $\|\pi - \sigma\| = \|\tilde{\pi} - \tilde{\sigma}\| = 2\|\tilde{\pi}\| = 2\|\tilde{\sigma}\|$. Furthermore,

$$\begin{aligned} \|\pi P - \sigma P\| &= \|\tilde{\pi} P - \tilde{\sigma} P\| \\ &= \sum_j \left\| \sum_i \tilde{\pi}_i P_{ij} - \sum_k \tilde{\sigma}_k P_{kj} \right\| \\ &= (1/\|\tilde{\pi}\|) \sum_j \left| \sum_{i,k} \tilde{\pi}_i \tilde{\sigma}_k (P_{ij} - P_{kj}) \right| \\ &\leq (1/\|\tilde{\pi}\|) \sum_{i,k} \tilde{\pi}_i \tilde{\sigma}_k \sum_j |P_{ij} - P_{kj}| \\ &\leq \|\tilde{\pi}\| (2 - 2\delta(P)) = \|\pi - \sigma\| (1 - \delta(P)), \end{aligned}$$

which proves the first part of the proposition. Iterating the inequality just proved yields that

$$\|\pi P^l - \sigma P^l\| \leq (1 - \delta)^l \|\pi - \sigma\| \leq 2(1 - \delta)^l. \quad (1.28)$$

This inequality for $\sigma = \pi P^n$ yields that $\|\pi P^l - \pi P^{l+n}\| \leq 2(1 - \delta)^l$. Thus the sequence πP^l is a Cauchy sequence and has a limit π^∞ , and $\pi^\infty P = \pi^\infty$. Finally, taking σ in (1.28) equal to π^∞ yields the last part of the proposition. \blacksquare

Proposition 1.7.3 typically does not yield the exact asymptotic rate that $\|\pi^l - \pi^\infty\|$ tends to zero. The asymptotic behavior can be investigated by computing $(I - zP)^{-1}$, and then matching powers of z in the identity $(I - zP)^{-1} = \sum_{n=0}^{\infty} z^n P^n$.

1.7.3 Continuous Time

Define, for $i \in \mathcal{S}$, $\tau_i^o = \min\{t > 0 : X(t) \neq i\}$, and $\tau_i = \min\{t > \tau_i^o : X(t) = i\}$. Thus, if $X(0) = i$, τ_i is the first time the process returns to state i , with the exception that $\tau_i = +\infty$ if the process never returns to state i . The following definitions are the same as when X is a discrete-time process. Let $M_i = E[\tau_i | X(0) = i]$. If $P[\tau_i < +\infty] < 1$, then state i is called *transient*.

Otherwise $P[\tau_i < +\infty] = 1$, and i is then said to be *positive - recurrent* if $M_i < +\infty$ and to be *null - recurrent* if $M_i = +\infty$.

The following propositions are analogous to those for discrete-time Markov processes. Proofs can be found in [1, 32].

Proposition 1.7.4 *Suppose X is irreducible.*

- (a) *All states are transient, or all are positive-recurrent, or all are null-recurrent.*
- (b) *For any initial distribution $\pi(0)$, $\lim_{t \rightarrow +\infty} \pi_i(t) = 1/(-q_{ii}M_i)$, with the understanding that the limit is zero if $M_i = +\infty$.*

Proposition 1.7.5 *Suppose X is irreducible and nonexplosive.*

- (a) *A probability distribution π is an equilibrium distribution if and only if $\pi Q = 0$.*
- (b) *An equilibrium probability distribution exists if and only if all states are positive-recurrent.*
- (c) *If all states are positive-recurrent, the equilibrium probability distribution is given by $\pi_i = 1/(-q_{ii}M_i)$. (In particular, if it exists, the equilibrium probability distribution is unique).*

Once again, the renewal theorem leads to an understanding of the convergence result of part (b) in Proposition 1.7.4. Let T_k denote the amount of time between the k th and $k + 1$ th visits to state i . Then the variables $T_k : k \geq 0$ are independent and identically distributed, and hence the sequence of visiting times is a renewal sequence. Let $H(t)$ denote the expected number of visits to state i during the interval $[0, t]$, so that H is the cumulative mean renewal function for the sequence T_1, T_2, \dots . As above, note that $M_i = E[T_k]$.

The renewal theorem is applied next to show that

$$\lim_{t \rightarrow \infty} P[X(t) = i] = 1/(-q_{ii}M_i). \quad (1.29)$$

Indeed, note that the definition of H , the fact that holding times for state i are exponentially distributed with parameter q_{ii} , integration by parts, a change of variable, and the renewal theorem, imply that

$$\begin{aligned} P[X(t) = i] &= P[X(t) = i, T_1 > t] + \int_0^t P[X(t) = i, (\text{last renewal before } t) \in ds] \\ &= P[X(0) = i] \exp(q_{ii}t) + \int_0^t \exp(q_{ii}(t-s)) dH(s) \\ &= P[X(0) = i] \exp(q_{ii}t) + \int_0^t \exp(q_{ii}(t-s)) (-q_{ii}) \{H(t) - H(s)\} ds \\ &= P[X(0) = i] \exp(q_{ii}t) + \int_0^t \exp(q_{ii}u) (-q_{ii}) \{H(t) - H(t-u)\} du \\ &\rightarrow \int_0^\infty \exp(q_{ii}u) (-q_{ii}) (u/M_i) du = 1/(-q_{ii}M_i). \end{aligned}$$

This establishes (1.29).

An immediate consequence of (1.29) is that if there is an equilibrium distribution π , then the Markov process is positive recurrent and the equilibrium distribution is given by

$$\pi_i = 1/(-q_{ii}M_i). \quad (1.30)$$

Conversely, if the process is positive recurrent, then there is an equilibrium probability distribution and it is given by (1.30).

Part (a) of Proposition 1.7.5 is suggested by the Kolmogorov forward equation $\partial\pi(t)/\partial t = \pi(t)Q$. The assumption that X be nonexplosive is needed (per homework problem), but the following proposition shows that the Markov processes encountered in most applications are nonexplosive.

Proposition 1.7.6 *Suppose X is irreducible. Fix a state i_o and for $k \geq 1$ let \mathcal{S}_k denote the set of states reachable from i in k jumps. Suppose for each $k \geq 1$ there is a constants γ_k so that the jump intensities on \mathcal{S}_k are bounded by γ_k , that is, suppose $-q_{ii} \leq \gamma_k$ for $i \in \mathcal{S}_k$. If $\sum_{k=1}^{\infty} \frac{1}{\gamma_k} = +\infty$, then the process X is nonexplosive.*

The proposition is proved after the following lemma is proved:

Lemma 1.7.7 *Suppose S is a random variable such that $E[S] \geq 1$ and S is the sum of finitely many independent exponentially distributed random variables (with possibly different means). Then $P[S \geq \frac{1}{3}] \geq \frac{1}{4}$.*

Proof. Let $S = T_1 + \dots + T_n$, such that the T_i are independent, exponentially distributed, and suppose $E[S] \geq 1$. If $E[T_j] \geq \frac{1}{3}$ for some j then $P[S \geq \frac{1}{3}] \geq P[T_j \geq \frac{1}{3}] \geq e^{-1} \geq \frac{1}{3}$. So it remains to prove the lemma in case $E[T_i] \leq \frac{1}{3}$ for all i . But then $Var(T_i) = E[T_i]^2 \leq \frac{1}{3}E[T_i]$. Therefore, $Var(S) \leq \frac{E(S)}{3}$. Thus, by the Chebychev inequality, $P[S \leq \frac{1}{3}] \leq P[|S - E[S]| \geq \frac{2E[S]}{3}] \leq \frac{9Var(S)}{4E[S]^2} \leq \frac{3}{4E[S]} \leq \frac{3}{4}$. The lemma is proved. ■

Proof. Let X^J denote the jump process and T_0, T_1, \dots the sequence of holding times of X . Select an increasing sequence of integers $k_0 = 0 < k_1 < k_2 < \dots$ so that $\sum_{k=k_{i-1}}^{k_i-1} \frac{1}{\gamma_k} \geq 1$ for all $i \geq 1$. Fix any particular sample path $(x^J(n))$ for the jump process with $x^J(0) = i_o$. Conditioned on the process X^J following this sample path, the holding times are independent, exponentially distributed, and $E[T_k] \geq \frac{1}{\gamma_k}$ for all $k \geq 0$. Thus, if S_i is the random variable $S_i = \sum_{k=k_{i-1}}^{k_i-1} T_k$, it follows from Lemma 1.7.7 that $P[S_i \geq \frac{1}{3}] \geq \frac{1}{4}$ for all i . Thus, for a suitable threshold $\tau_i \geq \frac{1}{3}$, the random variable $\Theta_i = I_{\{S_i \geq \tau_i\}}$ is a $Be(\frac{1}{4})$ random variable, $S_i \geq \frac{\Theta_i}{3}$ for all i , and the Θ_i 's are independent. Thus, $\sum_{k=0}^{k_j-1} T_k \geq \sum_{i=1}^j \Theta_i$, and by the law of large number, $\sum_{i=1}^j \Theta_i \xrightarrow{a.s.} \infty$ as $j \rightarrow \infty$. Thus, the sum of the holding times is a.s. infinite, so X is nonexplosive. ■

1.8 Classification of Birth-Death Processes

The classification of birth-death processes is relatively simple. Recall that a birth-death process with birth rates $(\lambda_k : k \geq 0)$ and death rates $(\mu_k : k \geq 1)$ is a pure-jump Markov process with state space equal to the set of nonnegative integers \mathbb{Z}_+ (with the appended state Δ , which is reached if

and only if infinitely many jumps occur in finite time), and generator matrix $Q = (q_{ij} : i, j \geq 0)$ given by

$$q_{ij} = \begin{cases} \lambda_i & \text{if } j = i + 1 \\ \mu_i & \text{if } j = i - 1 \\ -\lambda_i - \mu_i & \text{if } j = i \geq 1 \\ -\lambda_i & \text{if } j = i = 1 \\ 0 & \text{else.} \end{cases} \quad (1.31)$$

Suppose that the birth and death rates are all strictly positive. Then the process is irreducible. Let b_{in} denote the probability that, given the process starts in state i , it reaches state n without first reaching state 0. Clearly $b_{0n} = 0$ and $b_{nn} = 1$. Fix i with $1 \leq i \leq n - 1$, and derive an expression for b_{in} by first conditioning on the state reached upon the first jump of the process, starting from state i . By the analysis of jump probabilities, the probability the first jump is up is $\lambda_i/(\lambda_i + \mu_i)$ and the probability the first jump is down is $\mu_i/(\lambda_i + \mu_i)$. Thus,

$$b_{in} = \frac{\lambda_i}{\lambda_i + \mu_i} b_{i+1,n} + \frac{\mu_i}{\lambda_i + \mu_i} b_{i-1,n},$$

which can be rewritten as $\mu_i(b_{in} - b_{i-1,n}) = \lambda_i(b_{i+1,n} - b_{i,n})$. In particular, $b_{2n} - b_{1n} = b_{1n}\mu_1/\lambda_1$ and $b_{3n} - b_{2n} = b_{1n}\mu_1\mu_2/(\lambda_1\lambda_2)$, and so on, which upon summing yields the expression

$$b_{kn} = b_{1n} \sum_{i=0}^{k-1} \frac{\mu_1\mu_2 \dots \mu_i}{\lambda_1\lambda_2 \dots \lambda_i}.$$

with the convention that the $i = 0$ term in the sum is zero. Finally, the condition $b_{nn} = 1$ yields the solution

$$b_{1n} = \frac{1}{\sum_{i=0}^{n-1} \frac{\mu_1\mu_2 \dots \mu_i}{\lambda_1\lambda_2 \dots \lambda_i}}. \quad (1.32)$$

Note that b_{1n} is the probability, for initial state 1, of the event B_n that state n is reached without an earlier visit to state 0. Since $B_{n+1} \subset B_n$ for all $n \geq 1$,

$$P[\cap_{n \geq 1} B_n | X(0) = 1] = \lim_{n \rightarrow \infty} b_{1n} = 1/S_2 \quad (1.33)$$

where $S_2 = \sum_{i=0}^{\infty} \mu_1\mu_2 \dots \mu_i/\lambda_1\lambda_2 \dots \lambda_i$, with the understanding that the $i = 0$ term in the sum defining S_2 is one. Due to the definition of pure jump processes used, whenever X visits a state in \mathcal{S} the number of jumps up until that time is finite. Thus, on the event $\cap_{n \geq 1} B_n$, state zero is never reached. Conversely, if state zero is never reached, then either the process remains bounded (which has probability zero) or $\cap_{n \geq 1} B_n$ is true. Thus, $P[\text{zero is never reached} | X(0) = 1] = 1/S_2$. Consequently, X is recurrent if and only if $S_2 = \infty$.

Next, investigate the case $S_2 = \infty$. Then the process is recurrent, so that in particular explosion has probability zero, so that the process is positive recurrent if and only if $\pi Q = 0$.

Now $\pi Q = 0$ if and only if flow balance holds for any state k :

$$(\lambda_k + \mu_k)\pi_k = \lambda_{k-1}\pi_{k-1} + \mu_{k+1}\pi_{k+1}. \quad (1.34)$$

Equivalently, flow balance must hold for all sets of the form $\{0, \dots, n-1\}$, yielding that $\pi_{n-1}\lambda_{n-1} = \pi_n\lambda_n$, or $\pi_n = \pi_0\lambda_0 \dots \lambda_{n-1}/(\mu_1 \dots \mu_n)$. Thus, a probability distribution π with $\pi Q = 0$ exists if and only if $S_1 < +\infty$, where

$$S_1 = \sum_{n=0}^{\infty} \frac{\lambda_0 \dots \lambda_{n-1}}{\mu_1 \dots \mu_n}, \quad (1.35)$$

with the understanding that the $i = 0$ term in the sum defining S_1 is one. Still assuming $S_2 = \infty$, such a probability distribution exists if and only if the Markov process is positive recurrent.

In summary, the following proposition is proved.

Proposition 1.8.1 *If $S_2 < +\infty$ then all the states in \mathbb{Z}_+ are transient, and the process may be explosive. If $S_2 = +\infty$ and $S_1 = +\infty$ then all states are null-recurrent. If $S_2 = +\infty$ and $S_1 < +\infty$ then all states are positive-recurrent and the equilibrium probability distribution is given by $\pi_n = (\lambda_0 \dots \lambda_{n-1}) / (S_1 \mu_1 \dots \mu_n)$.*

Discrete-time birth-death processes have a similar characterization. They are discrete-time, time-homogeneous Markov processes with state space equal to the set of nonnegative integers. Let nonnegative birth probabilities $(\lambda_k : k \geq 0)$ and death probabilities $(\mu_k : k \geq 1)$ satisfy $\lambda_0 \leq 1$, and $\lambda_k + \mu_k \leq 1$ for $k \geq 1$. The one-step transition probability matrix $P = (p_{ij} : i, j \geq 0)$ is given by

$$p_{ij} = \begin{cases} \lambda_i & \text{if } j = i + 1 \\ \mu_i & \text{if } j = i - 1 \\ 1 - \lambda_i - \mu_i & \text{if } j = i \geq 1 \\ 1 - \lambda_i & \text{if } j = i = 0 \\ 0 & \text{else.} \end{cases} \quad (1.36)$$

Implicit in the specification of P is that births and deaths can't happen simultaneously. If the birth and death probabilities are strictly positive, Proposition 1.8.1 holds as before, with the exception that the discrete-time process cannot be explosive.³

1.9 Time Averages vs. Statistical Averages

Let X be a positive recurrent, irreducible, time-homogeneous Markov process with equilibrium probability distribution π . To be definite, suppose X is a continuous-time process, with pure-jump sample paths and generator matrix Q . The results of this section apply with minor modifications to the discrete-time setting as well. Above it is shown that renewal theory implies that $\lim_{t \rightarrow \infty} \pi_i(t) = \pi_i = 1 / (-q_{ii} M_i)$, where M_i is the mean "cycle time" of state i . A related consideration is convergence of the *empirical* distribution of the Markov process, where the empirical distribution is the distribution observed over a (usually large) time interval.

For a fixed state i , the fraction of time the process spends in state i during $[0, t]$ is

$$\frac{1}{t} \int_0^t I_{\{X(s)=i\}} ds$$

Let T_0 denote the time that the process is first in state i , and let T_k for $k \geq 1$ denote the time that the process jumps to state i for the k th time after T_0 . The cycle times $T_{k+1} - T_k$, $k \geq 0$ are independent and identically distributed, with mean M_i . Therefore, by the law of large numbers, with probability one,

$$\lim_{k \rightarrow \infty} T_k / k = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{l=0}^{k-1} (T_{l+1} - T_l) \quad (1.37)$$

$$= M_i \quad (1.38)$$

³If in addition $\lambda_i + \mu_i = 1$ for all i , then the discrete time process has period 2.

Furthermore, during the k th cycle interval $[T_k, T_{k+1})$, the amount of time spent by the process in state i is exponentially distributed with mean $-1/q_{ii}$, and the time spent in the state during disjoint cycles is independent. Thus, with probability one,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_0^{T_k} I_{\{X(s)=i\}} dt = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{l=0}^{k-1} \int_{T_l}^{T_{l+1}} I_{\{X(s)=i\}} ds \quad (1.39)$$

$$= E \left[\int_{T_0}^{T_1} I_{\{X(s)=i\}} ds \right] \quad (1.40)$$

$$= -1/q_{ii} \quad (1.41)$$

Combining these two observations yields that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I_{\{X(s)=i\}} ds = 1/(-q_{ii}M_i) = \pi_i \quad (1.42)$$

with probability one. In short, the limit (1.42) is expected, because the process spends on average $-1/q_{ii}$ time units in state i per cycle from state i , and the cycle rate is $1/M_i$. Of course, since state i is arbitrary, if j is any other state then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I_{\{X(s)=j\}} ds = 1/(-q_{jj}M_j) = \pi_j \quad (1.43)$$

By considering how the time in state j is distributed among the cycles from state i , it follows that the mean time spent in state j per cycle from state i is $M_i\pi_j$.

So for any nonnegative function ϕ on \mathcal{S} ,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \phi(X(s)) ds &= \lim_{k \rightarrow \infty} \frac{1}{kM_i} \int_0^{T_k} \phi(X(s)) ds \\ &= \frac{1}{M_i} E \left[\int_{T_0}^{T_1} \phi(X(s)) ds \right] \\ &= \frac{1}{M_i} E \left[\sum_{j \in \mathcal{S}} \phi(j) \int_{T_0}^{T_1} I_{\{X(s)=j\}} ds \right] \\ &= \frac{1}{M_i} \sum_{j \in \mathcal{S}} \phi(j) E \left[\int_{T_0}^{T_1} I_{\{X(s)=j\}} ds \right] \\ &= \sum_{j \in \mathcal{S}} \phi(j) \pi_j \end{aligned} \quad (1.44)$$

Finally, if ϕ is a function of \mathcal{S} such that either $\sum_{j \in \mathcal{S}} \phi_+(j)\pi_j < \infty$ or $\sum_{j \in \mathcal{S}} \phi_-(j)\pi_j < \infty$, then since (1.44) holds for both ϕ_+ and ϕ_- , it must hold for ϕ itself.

1.10 Queueing Systems, M/M/1 Queue and Little's Law

Some basic terminology of queueing theory will now be explained. A simple type of queueing system is pictured in Figure 1.13. Notice that the *system* is comprised of a *queue* and a *server*. Ordinarily

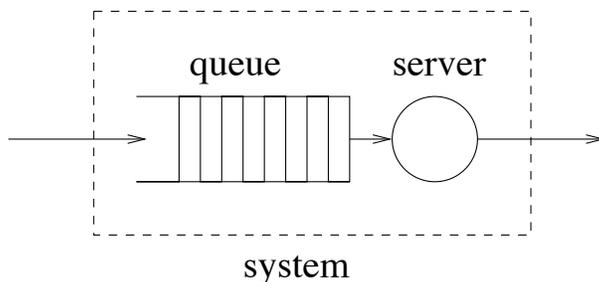


Figure 1.13: A single server queueing system

whenever the system is not empty, there is a customer in the server, and any other customers in the system are waiting in the queue. When the service of a customer is complete it departs from the server and then another customer from the queue, if any, immediately enters the server. The choice of which customer to be served next depends on the *service discipline*. Common service disciplines are first-come first-served (FCFS) in which customers are served in the order of their arrival, or last-come first-served (LCFS) in which the customer that arrived most recently is served next. Some of the more complicated service disciplines involve priority classes, or the notion of “processor sharing” in which all customers present in the system receive equal attention from the server.

Often models of queueing systems involve a stochastic description. For example, given positive parameters λ and μ , we may declare that the arrival process is a Poisson process with rate λ , and that the service times of the customers are independent and exponentially distributed with parameter μ . Many queueing systems are given labels of the form A/B/s, where “A” is chosen to denote the type of arrival process, “B” is used to denote the type of departure process, and s is the number of servers in the system. In particular, the system just described is called an M/M/1 queueing system, so-named because the arrival process is memoryless (i.e. a Poisson arrival process), the service times are memoryless (i.e. are exponentially distributed), and there is a single server. Other labels for queueing systems have a fourth descriptor and thus have the form A/B/s/b, where b denotes the maximum number of customers that can be in the system. Thus, an M/M/1 system is also an M/M/1/ ∞ system, because there is no finite bound on the number of customers in the system.

A second way to specify an M/M/1 queueing system with parameters λ and μ is to let $A(t)$ and $D(t)$ be independent Poisson processes with rates λ and μ respectively. Process A marks customer arrival times and process D marks *potential* customer departure times. The number of customers in the system, starting from some initial value $N(0)$, evolves as follows. Each time there is a jump of A , a customer arrives to the system. Each time there is a jump of D , there is a potential departure, meaning that if there is a customer in the server at the time of the jump then the customer departs. If a potential departure occurs when the system is empty then the potential departure has no effect on the system. The number of customers in the system N can thus be expressed as

$$N(t) = N(0) + A(t) + \int_0^t I_{\{N(s^-) \geq 1\}} dD(s)$$

It is easy to verify that the resulting process N is Markov, which leads to the third specification of an M/M/1 queueing system.

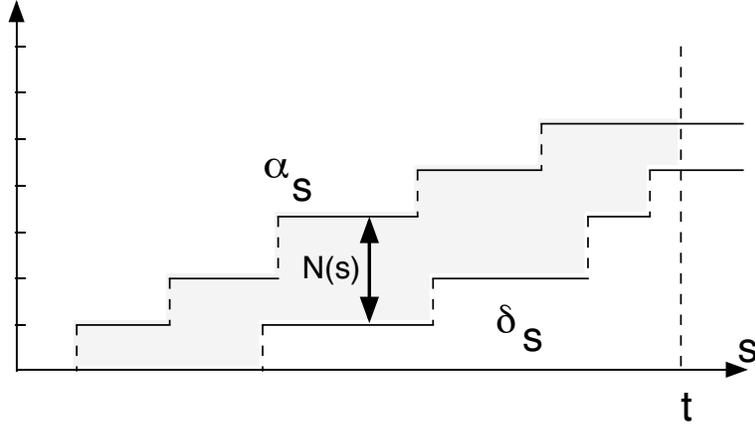


Figure 1.14: A single server queueing system

A third way to specify an M/M/1 queueing system is that the number of customers in the system $N(t)$ is a birth-death process with $\lambda_k = \lambda$ and $\mu_k = \mu$ for all k , for some parameters λ and μ . Let $\rho = \lambda/\mu$. Using the classification criteria derived for birth-death processes, it is easy to see that the system is recurrent if and only if $\rho \leq 1$, and that it is positive recurrent if and only if $\rho < 1$. Moreover, if $\rho < 1$ then the equilibrium distribution for the number of customers in the system is given by $\pi_k = (1 - \rho)\rho^k$ for $k \geq 0$. This is the geometric distribution with zero as a possible value, and with mean

$$\bar{N} = \sum_{k=0}^{\infty} k\pi_k = (1 - \rho)\rho \sum_{k=1}^{\infty} \rho^{k-1}k = (1 - \rho)\rho \left(\frac{1}{1 - \rho}\right)' = \frac{\rho}{1 - \rho}$$

The probability the server is busy, which is also the mean number of customers in the server, is $1 - \pi_0 = \rho$. The mean number of customers in the queue is thus given by $\rho/(1 - \rho) - \rho = \rho^2/(1 - \rho)$. This third specification is the most commonly used way to define an M/M/1 queueing process.

Since the M/M/1 process $N(t)$ is positive recurrent, the Markov ergodic convergence theorem implies that the statistical averages just computed, such as \bar{N} , are also equal to the limit of the time-averaged number of customers in the system as the averaging interval tends to infinity.

An important performance measure for a queueing system is the mean time spent in the system or the mean time spent in the queue. Little's law, described next, is a quite general and useful relationship that aids in computing mean transit time.

Little's law can be applied in a great variety of circumstances involving flow through a system with delay. In the context of queueing systems we speak of a flow of customers, but the same principle applies to a flow of water through a pipe. Little's law is that $\lambda T = \bar{N}$ where λ is the mean flow rate, T is the mean delay in the system, and \bar{N} is the mean content of the queue. For example, if water flows through a pipe with volume one cubic meter at the rate of two cubic meters per minute, then the mean time (averaged over all drops of water) that water spends in the pipe is $T = \bar{N}/\lambda = 1/2$ minute. This is clear if water flows through the pipe without mixing, for then the transit time of each drop of water is 1/2 minute. However, mixing within the pipe does not effect the average transit time.

Little's law is actually a set of results, each with somewhat different mathematical assumptions.

The following version is quite general. Figure 1.14 pictures the cumulative number of arrivals ($\alpha(t)$) and the cumulative number of departures ($\delta(t)$) versus time, for a queueing system assumed to be initially empty. Note that the number of customers in the system at any time s is given by the difference $N(s) = \alpha(s) - \delta(s)$, which is the vertical distance between the arrival and departure graphs in the figure. On the other hand, assuming that customers are served in first-come first-served order, the horizontal distance between the graphs gives the times in system for the customers. Given a (usually large) $t > 0$, let γ_t denote the area of the region between the two graphs over the interval $[0, t]$. This is the shaded region indicated in the figure. It is then natural to define the time-averaged values of arrival rate and system content as

$$\bar{\lambda}_t = \alpha(t)/t \quad \text{and} \quad \bar{N}_t = \frac{1}{t} \int_0^t N(s) ds = \gamma_t/t$$

Finally, the average, over the $\alpha(t)$ customers that arrive during the interval $[0, t]$, of the time spent in the system up to time t , is given by

$$\bar{T}_t = \gamma_t/\alpha(t).$$

Once these definitions are accepted, we have the obvious simple relation:

$$\bar{N}_t = \bar{\lambda}_t \bar{T}_t \tag{1.45}$$

Consider next (1.45) in the limit as $t \rightarrow \infty$. If any two of the three variables in (1.45) converge to finite positive quantities, then so does the third variable, and in the limit Little's law is obtained.

For example, the number of customers in an M/M/1 queue is a positive recurrent Markov process so that

$$\lim_{t \rightarrow \infty} \bar{N}_t = \bar{N} = \rho/(1 - \rho)$$

where calculation of the statistical mean \bar{N} was previously discussed. Also, by the weak renewal theorem or the law of large numbers applied to interarrival times, we have that the Poisson arrival process for an M/M/1 queue satisfies $\lim_{t \rightarrow \infty} \bar{\lambda}_t = \lambda$ with probability one. Thus,

$$\lim_{t \rightarrow \infty} \bar{T}_t = \bar{N}/\lambda = \frac{1}{\mu - \lambda}.$$

In this sense, the average waiting time in an M/M/1 system is $1/(\mu - \lambda)$. The average time in service is $1/\mu$ (this follows from the third description of an M/M/1 queue, or also from Little's law applied to the server alone) so that the average waiting time in queue is given by $W = 1/(\mu - \lambda) - 1/\mu = \rho/(\mu - \lambda)$. This final result also follows from Little's law applied to the queue alone.

1.11 Mean Arrival Rate, Distributions Seen by Arrivals, and PASTA

The mean arrival rate for the M/M/1 system is λ , the parameter of the Poisson arrival process. However for some queueing systems the arrival rate depends on the number of customers in the system. In such cases the mean arrival rate is still typically meaningful, and it can be used in Little's law.

Suppose the number of customers in a queueing system is modeled by a birth death process with arrival rates (λ_k) and departure rates (μ_k). Suppose in addition that the process is positive

recurrent. Intuitively, the process spends a fraction of time π_k in state k and while in state k the arrival rate is λ_k . Therefore, the average arrival rate is

$$\bar{\lambda} = \sum_{k=0}^{\infty} \pi_k \lambda_k$$

Similarly the average departure rate is

$$\bar{\mu} = \sum_{k=1}^{\infty} \pi_k \mu_k$$

and of course $\bar{\lambda} = \bar{\mu}$ because both are equal to the throughput of the system.

Often the distribution of a system at particular system-related sampling times are more important than the distribution in equilibrium. For example, the distribution seen by arriving customers may be the most relevant distribution, as far as the customers are concerned. If the arrival rate depends on the number of customers in the system then the distribution seen by arrivals need not be the same as the equilibrium distribution. Intuitively, $\pi_k \lambda_k$ is the long-term frequency of arrivals which occur when there are k customers in the system, so that the fraction of customers that see k customers in the system upon arrival is given by

$$r_k = \frac{\pi_k \lambda_k}{\bar{\lambda}}.$$

The following is an example of a system with variable arrival rate.

Example 1.5 (Single-server, discouraged arrivals)

Suppose $\lambda_k = \alpha/(k+1)$ and $\mu_k = \mu$ for all k , where μ and α are positive constants. Then

$$S_2 = \sum_{k=0}^{\infty} \frac{(k+1)! \mu^k}{\alpha^k} = \infty \quad \text{and} \quad S_1 = \sum_{k=0}^{\infty} \frac{\alpha^k}{k! \mu^k} = \exp\left(\frac{\alpha}{\mu}\right) < \infty$$

so that the number of customers in the system is a positive recurrent Markov process, with no additional restrictions on α and μ . Moreover, the equilibrium probability distribution is given by $\pi_k = (\alpha/\mu)^k \exp(-\alpha/\mu)/k!$, which is the Poisson distribution with mean $\bar{N} = \alpha/\mu$. The mean arrival rate is

$$\bar{\lambda} = \sum_{k=0}^{\infty} \frac{\pi_k \alpha}{k+1} = \mu \exp\left(-\frac{\alpha}{\mu}\right) \sum_{k=0}^{\infty} \frac{(\alpha/\mu)^{k+1}}{(k+1)!} = \mu \exp\left(-\frac{\alpha}{\mu}\right) (\exp\left(\frac{\alpha}{\mu}\right) - 1) = \mu(1 - \exp(-\frac{\alpha}{\mu})). \quad (1.46)$$

This expression derived for $\bar{\lambda}$ is clearly equal to $\bar{\mu}$, because the departure rate is μ with probability $1 - \pi_0$ and zero otherwise. The distribution of the number of customers in the system seen by arrivals, (r_k) is given by

$$r_k = \frac{\pi_k \alpha}{(k+1)\bar{\lambda}} = \frac{(\alpha/\mu)^{k+1} \exp(-\alpha/\mu)}{(k+1)!(1 - \exp(-\alpha/\mu))} \quad \text{for } k \geq 0$$

which in words can be described as the result of removing the probability mass at zero in the Poisson distribution, shifting the distribution down by one, and then renormalizing. The mean number of customers in the queue seen by a typical arrival is therefore $(\alpha/\mu - 1)/(1 - \exp(-\alpha/\mu))$.

This mean is somewhat less than \bar{N} because, roughly speaking, the customer arrival rate is higher when the system is more lightly loaded.

The equivalence of time-averages and statistical averages for computing the mean arrival rate and the distribution seen by arrivals can be shown by application of ergodic properties of the processes involved. The associated formal approach is described next, in slightly more generality. Let X denote an irreducible, positive-recurrent pure-jump Markov process. If the process makes a jump from state i to state j at time t , say that a transition of type (i, j) occurs. The sequence of transitions of X forms a new Markov process, Y . The process Y is a discrete-time Markov process with state space $\{(i, j) \in \mathcal{S} \times \mathcal{S} : q_{ij} > 0\}$, and it can be described in terms of the jump process for X , by $Y(k) = (X^J(k-1), X^J(k))$ for $k \geq 0$. (Let $X^J(-1)$ be defined arbitrarily.)

The one-step transition probability matrix of the jump process X^J is given by $\pi_{ij}^J = q_{ij}/(-q_{ii})$, and X^J is recurrent because X is recurrent. Its equilibrium distribution π^J (if it exists) is proportional to $-\pi_i q_{ii}$ (see problem set 1), and X^J is *positive* recurrent if and only if this distribution can be normalized to make a probability distribution, i.e. if and only if $R = -\sum_i \pi_i q_{ii} < \infty$. Assume for simplicity that X^J is positive recurrent. Then $\pi_i^J = -\pi_i q_{ii}/R$ is the equilibrium probability distribution of X^J . Furthermore, Y is positive recurrent and its equilibrium distribution is given by

$$\begin{aligned} \pi_{ij}^Y &= \pi_i^J p_{ij}^J \\ &= \frac{-\pi_i q_{ii}}{R} \frac{q_{ij}}{-q_{ii}} \\ &= \frac{\pi_i q_{ij}}{R} \end{aligned}$$

Since limiting time averages equal statistical averages for Y ,

$$\lim_{n \rightarrow \infty} (\text{number of first } n \text{ transitions of } X \text{ that are type } (i, j))/n = \pi_i q_{ij}/R$$

with probability one. Therefore, if $A \subset \mathcal{S} \times \mathcal{S}$, and if $i \in A$, then

$$\lim_{n \rightarrow \infty} \frac{\text{number of first } n \text{ transitions of } X \text{ that are type } (i, j)}{\text{number of first } n \text{ transitions of } X \text{ with type in } A} = \frac{\pi_i q_{ij}}{\sum_{(i', j') \in A} \pi_{i'} q_{i' j'}} \quad (1.47)$$

We remark (1.47) is still true if the assumption $R < \infty$ is replaced by the weaker assumption that the denominator on the righthand side of (1.47) is finite. In case $R = \infty$ the statement is then an instance of a ratio limit theorem for recurrent (not necessarily positive recurrent) Markov processes.

To apply this setup to the special case of a queueing system in which the number of customers in the system is a Markov birth-death processes, let the set A be the set of transitions of the form $(i, i+1)$. Then deduce that the fraction of the first n arrivals that see i customers in the system upon arrival converges to $\pi_i \lambda_i / \sum_j \pi_j \lambda_j$ with probability one.

Note that if $\lambda_i = \lambda$ for all i , then $\bar{\lambda} = \lambda$ and $\pi = r$. The condition $\lambda_i = \lambda$ also implies that the arrival process is Poisson. This situation is called ‘‘Poisson Arrivals See Time Averages’’ (PASTA).

1.12 More Examples of Queueing Systems Modeled as Markov Birth-Death Processes

For each of the four examples of this section it is assumed that new customers are offered to the system according to a Poisson process with rate λ , so that the PASTA property holds. Also, when

there are k customers in the system then the service rate is μ_k for some given numbers μ_k . The number of customers in the system is a Markov birth-death process with $\lambda_k = \lambda$ for all k . Since the number of transitions of the process up to any given time t is at most twice the number of customers that arrived by time t , the Markov process is not explosive. Therefore the process is positive recurrent if and only if S_1 is finite, where

$$S_1 = \sum_{k=0}^{\infty} \frac{\lambda^k}{\mu_1 \mu_2 \cdots \mu_k}$$

Special cases of this example are presented in the next four examples.

Example 1.6 (M/M/ m systems) An M/M/ m queueing system consists of a single queue and m servers. The arrival process is Poisson with some rate λ and the customer service times are independent and exponentially distributed with mean μ for some $\mu > 0$. The total number of customers in the system is a birth-death process with $\mu_k = \mu \min(k, m)$. Let $\rho = \lambda/(m\mu)$. Since $\mu_k = m\mu$ for all k large enough it is easy to check that the process is positive recurrent if and only if $\rho < 1$. Assume now that $\rho < 1$. Then the equilibrium distribution is given by

$$\begin{aligned} \pi_k &= \frac{(\lambda/\mu)^k}{S_1 k!} \quad \text{for } 0 \leq k \leq m \\ \pi_{m+j} &= \pi_m \rho^j \quad \text{for } j \geq 1 \end{aligned}$$

where S_1 is chosen to make the probabilities sum to one (use the fact $1 + \rho + \rho^2 \dots = 1/(1 - \rho)$):

$$S_1 = \left(\sum_{k=0}^{m-1} \frac{(\lambda/\mu)^k}{k!} \right) + \frac{(\lambda/\mu)^m}{m!(1 - \rho)}.$$

An arriving customer must join the queue (rather than go directly to a server) if and only if the system has m or more customers in it. By the PASTA property, this is the same as the equilibrium probability of having m or more customers in the system:

$$P_Q = \sum_{j=0}^{\infty} \pi_{m+j} = \pi_m / (1 - \rho)$$

This formula is called the *Erlang C* formula for probability of queueing.

Example 1.7 (M/M/ m/m systems) An M/M/ m/m queueing system consists of m servers. The arrival process is Poisson with some rate λ and the customer service times are independent and exponentially distributed with mean μ for some $\mu > 0$. Since there is no queue, if a customer arrives when there are already m customers in the system, then the arrival is blocked and cleared from the system. The total number of customers in the system is a birth death process, but with the state space reduced to $\{0, 1, \dots, m\}$, and with $\mu_k = k\mu$ for $1 \leq k \leq m$. The unique equilibrium distribution is given by

$$\pi_k = \frac{(\lambda/\mu)^k}{S_1 k!} \quad \text{for } 0 \leq k \leq m$$

where S_1 is chosen to make the probabilities sum to one.

An arriving customer is blocked and cleared from the system if and only if the system already has m customers in it. By the PASTA property, this is the same as the equilibrium probability of

having m customers in the system:

$$P_B = \pi_m = \frac{\frac{(\lambda/\mu)^m}{m!}}{\sum_{j=0}^m \frac{(\lambda/\mu)^j}{j!}}$$

This formula is called the *Erlang B* formula for probability of blocking.

Example 1.8 (A system with a discouraged server) The number of customers in this system is a birth-death process with constant birth rate λ and death rates $\mu_k = 1/k$. It is easy to check that all states are transient for any positive value of λ (to verify this it suffices to check that $S_2 < \infty$). It is not difficult to show that $N(t)$ converges to $+\infty$ with probability one as $t \rightarrow \infty$.

Example 1.9 (A barely stable system) The number of customers in this system is a birth-death process with constant birth rate λ and death rates $\mu_k = \frac{\lambda(1+k^2)}{1+(k-1)^2}$ for all $k \geq 1$. Since the departure rates are barely larger than the arrival rates, this system is near the borderline between recurrence and transience. However, we see that

$$S_1 = \sum_{k=0}^{\infty} \frac{1}{1+k^2} < \infty$$

so that $N(t)$ is positive recurrent with equilibrium distribution $\pi_k = 1/(S_1(1+k^2))$. Note that the mean number of customers in the system is

$$\bar{N} = \sum_{k=0}^{\infty} k/(S_1(1+k^2)) = \infty$$

By Little's law the mean time customers spend in the system is also infinity. It is debatable whether this system should be thought of as "stable" even though all states are positive recurrent and all waiting times are finite with probability one.

1.13 Method of Phases and Quasi Birth-Death Processes

The method of phases in a broad sense refers to approximating general probability distributions by sums and mixtures of exponential distributions. Such distributions can be obtained as the hitting time of a target state of a finite state Markov process. The state of the Markov process at a given time represents the "phase" of the distribution. The method permits Markov processes to model systems in which the original distributions are not exponential.

To illustrate the method here we consider the class of random processes called quasi birth and death (QBD) processes. These processes generalize M/M/1 queueing models in that the arrival and departure rates depend on the state of a phase process. (For more background, see M.F. Neuts, *Matrix-Geometric Solutions in Stochastic Models – An Algorithmic Approach*, The Johns Hopkins University Press, Baltimore, MD, 1981 and B. Hajek, *Birth-and-Death Processes on the Integers with Phases and General Boundaries*, Journal of Applied Probability, pp. 488-499, 1982.

Let k be a positive integer, denoting the number of possible phases. A QBD process is a time-homogeneous Markov process. The state space of the process is the set \mathcal{S} equal to $\{(l, \theta) : l \geq 1, 1 \leq \theta \leq k\}$. We call l the level and θ the phase of the state (l, θ) . Here we shall restrict our

attention to continuous time processes. Let A_0, A_1, A_2 , and A_{00} denote $k \times k$ matrices so that the matrix Q^* defined by

$$Q^* = \begin{pmatrix} A_{00} & A_2 & & & & \\ A_0 & A_1 & A_2 & & & \\ & A_0 & A_1 & A_2 & & \\ & & A_0 & A_1 & A_2 & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \end{pmatrix} \quad (1.48)$$

is a generator matrix for the state space \mathcal{S} . Equivalently, the matrices A_0 and A_2 have nonnegative entries, the matrices A_1 and A_{00} have nonnegative off-diagonal entries and nonpositive diagonal entries, the matrix Q defined by $Q = A_0 + A_1 + A_2$ and the matrix $A_{00} + A_2$ both have row sums equal to zero.

Suppose that Q^* and Q are both irreducible. Define π to be the equilibrium probability vector for Q , so that $\pi Q = 0$ and set $\mu = \pi A_2 e - \pi A_0 e$ where e is the k vector of all ones. Then all states are positive recurrent for Q^* if and only if $\mu < 0$. If $\mu < 0$ then the equilibrium distribution for Q^* is given by (x_0, x_1, \dots) where

$$x_0 = \pi_0 / c, \quad x_k = x_0 R^k, \quad c = \pi_0 (I - R)^{-1} e, \quad R = A_2 B \quad (1.49)$$

where π_0 is the equilibrium probability distribution for a k -state Markov process with generator matrix $A_{00} + A_{01} B A_{10}$, and B is the minimal positive solution to

$$B = -(A_1 + A_2 B A_0)^{-1} \quad (1.50)$$

An effective numerical procedure for finding B is to iterate equation (1.50). In other words, set $B^{(0)}$ to be the matrix of all zeros, and sequentially compute $B^{(n)}$ by $B^{(n)} = -(A_1 + A_2 B^{(n-1)} A_0)^{-1}$.

Exercises

1. Let (L_t, Θ_t) denote the QBD process. Under what conditions on A_0, A_1, A_2 and A_{00} is the process (Θ_t) a Markov process? Prove your answer.
2. Consider the following variation of an M/M/1 queue. There is a two state Markov process with states 1 and 2 and transition rates $q_{12} = 1$ and $q_{21} = 2$. The two state process is a phase process for the queue. When the phase is 1 there are no arrivals to the queue, and when the phase is 2 arrivals occur at an average rate of 2 customers per second, according to a Poisson process. Service times of all customers are independent and exponentially distributed with mean one. Suppose a QBD model is used. What values of the matrices A_0, A_1, A_2 and A_{00} should be used? Find (numerical solution OK) B, R , and the equilibrium distribution $(x_l)_{l \geq 0}$. Finally, describe the equilibrium probability distribution of the number of customers in the system. (Solution:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_1 = \begin{pmatrix} -2 & 1 \\ 2 & -5 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad (1.51)$$

$$A_{00} = \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \quad B = \begin{pmatrix} 0.8090 & 0.1910 \\ 0.6179 & 0.3819 \end{pmatrix} \quad R = \begin{pmatrix} 0 & 0 \\ 1.236 & 0.764 \end{pmatrix} \quad (1.52)$$

$x_k = (.764)^k (0.1273, 0.0787)$ for $k \geq 1$ and $x_0 = (0.255, 0.0787)$. In equilibrium, $P[L = l] = (0.206)(0.764)^l$ for $l \geq 1$ and $P[L = 0] = 1/3$. Thus, the distribution for L is a mixture of the geometric distribution and a distribution concentrated at zero.

1.14 Markov Fluid Model of a Queue

A limiting form of quasi birth and death processes is a queue with a Markov-modulated rate of arrival, which is briefly described in this section. See D. Mitra, “Stochastic Theory of a Fluid Model of Producers and Consumers Coupled by a Buffer”, *Annals of Applied Probability*, Vol. 20, pp. 646-676, 1988 for more information.

Let Q be the generator matrix of an irreducible Markov process with state space $\{1, \dots, m\}$. Suppose there is a buffer and that when the process is in state i , the net rate (arrival rate minus departure rate) of flow of fluid into the buffer is d_i . The only exception to this rule is that if the buffer is empty and the Markov process is in a state i such that $d_i \leq 0$ then the buffer simply remains empty. Let D denote the diagonal matrix with d_1, \dots, d_m along the diagonal. See Figure 1.15.

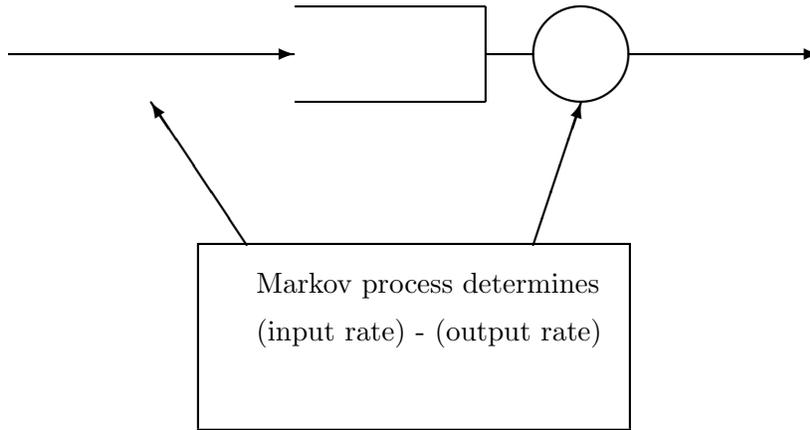


Figure 1.15: Fluid queue with net input rate determined by a Markov process

The average drift is defined to be $\sum_{i=1}^m w_i d_i$, or wDe , where w is the equilibrium probability distribution for the finite state Markov process and e is the column vector of all 1's.

Let L_t denote the level of fluid in the buffer at time t and let θ_t denote the state of the finite Markov process at time t . Then (L_t, θ_t) is a continuous state Markov process, and is a limiting form of a quasi birth-death process. In the remainder of this section we describe how to find the equilibrium distribution of this process.

Let $F(x, i, t) = P[\theta_t = i, L_t \leq x]$. Kolmogorov's forward equations are derived by beginning with the observation

$$F(x, i, t + h) = h \sum_{j=1}^m F(x, j, t) q_{j,i} + o(h) + F(x - hd_i, i, t). \quad (1.53)$$

Subtracting $F(x, i, t)$ from each side of this equation and letting $h \rightarrow 0$ yields

$$\frac{\partial F(x, i, t)}{\partial t} = \sum_{j=1}^m F(x, j, t) q_{j,i} - \frac{\partial F(x, i, t)}{\partial x} d_i \quad (1.54)$$

or in matrix notation, letting $F(t)$ denote the row vector $F(t) = (F(x, 1, t), \dots, F(x, m, t))$,

$$\frac{\partial F}{\partial t} = FQ - \frac{\partial F}{\partial x} D \quad (1.55)$$

Conditions on the equilibrium distribution $F(x, i)$ are found by setting $\frac{\partial F}{\partial t} = 0$, yielding the equation $FQ = \frac{\partial F}{\partial x}D$. By definition, we know $F(\infty, i) = w_i$. Motivated by this and the theory of linear systems, we seek a solution of the form

$$F(x, i) = w - \sum_{i=1}^k a_i \phi_i e^{xz_i} \quad (1.56)$$

where (ϕ_i, z_i) are eigenvalue-eigenvector pairs, meaning that $\phi_i Q = z_i \phi_i D$.

To avoid substantial technical complications, we make the assumption that Q is reversible, meaning that $w_i q_{i,j} = w_j q_{j,i}$ for $1 \leq i, j \leq m$. Then all the eigenvalues are real valued. Let $k = |\{i : d_i > 0\}|$. It can be shown that there are k strictly negative eigenvalues (counting multiplicity) and those are the ones to be included in the sum in (1.56). Finally, the k coefficients a_1, \dots, a_k can be found by satisfying the k boundary conditions:

$$F(0, i) = 0 \quad \text{whenever } d_i > 0 \quad (1.57)$$

For example, suppose

$$Q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.58)$$

We compute that $w = (\frac{2}{3}, \frac{1}{3})$ and that $(F(x, 1), F(x, 2)) = (\frac{2 - \exp(-x)}{3}, \frac{1 - \exp(-x)}{3})$. In particular, in equilibrium, $P[L_t \geq x] = \frac{2 \exp(-x)}{3}$.

1.15 Problems

1.1. Poisson merger

Summing counting processes corresponds to “merging” point processes. Show that the sum of K independent Poisson processes, having rates $\lambda_1, \dots, \lambda_K$, respectively, is a Poisson process with rate $\lambda_1 + \dots + \lambda_K$. (Hint: First formulate and prove a similar result for sums of random variables, and then think about what else is needed to get the result for Poisson processes. You can use any one of the equivalent definitions given by Proposition 1.5.1 in the notes. Don’t forget to check required independence properties.)

1.2. Poisson splitting

Consider a stream of customers modeled by a Poisson process, and suppose each customer is one of K types. Let (p_1, \dots, p_K) be a probability vector, and suppose that for each k , the k^{th} customer is type i with probability p_i . The types of the customers are mutually independent and also independent of the arrival times of the customers. Show that the stream of customers of a given type i is again a Poisson stream, and that its rate is λp_i . (Same hint as in the previous problem applies.) Show furthermore that the K substreams are mutually independent.

1.3. Poisson method for coupon collector’s problem

(a) Suppose a stream of coupons arrives according to a Poisson process $(A(t) : t \geq 0)$ with rate $\lambda = 1$, and suppose there are k types of coupons. (In network applications, the coupons could be pieces of a file to be distributed by some sort of gossip algorithm.) The type of each coupon in the

stream is randomly drawn from the k types, each possibility having probability $\frac{1}{k}$, and the types of different coupons are mutually independent. Let $p(k, t)$ be the probability that at least one coupon of each type arrives by time t . (The letter “ p ” is used here because the number of coupons arriving by time t has the Poisson distribution). Express $p(k, t)$ in terms of k and t .

(b) Find $\lim_{k \rightarrow \infty} p(k, k \ln k + kc)$ for an arbitrary constant c . That is, find limit of the probability that the collection is complete at time $t = k \ln k + kc$. (Hint: If $a_k \rightarrow a$ as $k \rightarrow \infty$, then $(1 + \frac{a_k}{k})^k \rightarrow e^a$.)

(c) The rest of this problem shows that the limit found in part (b) also holds if the total number of coupons is deterministic, rather than Poisson distributed. One idea is that if t is large, then $A(t)$ is not too far from its mean with high probability. Show, specifically, that

$$\lim_{k \rightarrow \infty} P[A(k \ln k + kc) \geq k \ln k + kc'] = \begin{cases} 0 & \text{if } c < c' \\ 1 & \text{if } c > c' \end{cases}$$

(d) Let $d(k, n)$ denote the probability that the collection is complete after n coupon arrivals. (The letter “ d ” is used here because the number of coupons, n , is deterministic.) Show that for any k, t , and n fixed, $d(k, n)P[A(t) \geq n] \leq p(k, t) \leq P[A(t) \geq n] + P[A(t) \leq n]d(k, n)$.

(e) Combine parts (c) and (d) to identify $\lim_{k \rightarrow \infty} d(k, k \ln k + kc)$.

1.4. The sum of a random number of random variables

Let $(X_i : i \geq 0)$ be a sequence of independent and identically distributed random variables. Let $S = X_1 + \dots + X_N$, where N is a nonnegative random variable that is independent of the sequence.

(a) Express $E[S]$ and $Var(S)$ in terms of the mean and variance of X_1 and N . (Hint: Use the fact $E[S^k] = E[E[S^k|N]]$.) (b) Express the characteristic function Φ_S in terms of the characteristic function Φ_X of X_1 and the z -transform $B(z)$ of N .

1.5. Mean hitting time for a simple Markov process

Let $(X(n) : n \geq 0)$ denote a discrete-time, time-homogeneous Markov chain with state space $\{0, 1, 2, 3\}$ and one-step transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 - a & 0 & a & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for some constant a with $0 \leq a \leq 1$. (a) Sketch the transition probability diagram for X and give the equilibrium probability vector. If the equilibrium vector is not unique, describe all the equilibrium probability vectors.

(b) Compute $E[\min\{n \geq 1 : X(n) = 3\} | X(0) = 0]$.

1.6. A two station pipeline in continuous time

Consider a pipeline consisting of two single-buffer stages in series. Model the system as a continuous-time Markov process. Suppose new packets are offered to the first stage according to a rate λ Poisson process. A new packet is accepted at stage one if the buffer in stage one is empty at the time of arrival. Otherwise the new packet is lost. If at a fixed time t there is a packet in stage one and no packet in stage two, then the packet is transferred during $[t, t + h)$ to stage two with probability $h\mu_1 + o(h)$. Similarly, if at time t the second stage has a packet, then the packet leaves the system during $[t, t + h)$ with probability $h\mu_2 + o(h)$, independently of the state of stage one. Finally, the probability of two or more arrival, transfer, or departure events during $[t, t + h)$ is $o(h)$. (a) What

is an appropriate state-space for this model? (b) Sketch a transition rate diagram. (c) Write down the Q matrix. (d) Derive the throughput, assuming that $\lambda = \mu_1 = \mu_2 = 1$. (e) Still assuming $\lambda = \mu_1 = \mu_2 = 1$. Suppose the system starts with one packet in each stage. What is the expected time until both buffers are empty?

1.7. Simple population growth models

A certain population of living cells begins with a single cell. Each cell present at a time t splits into two cells during the interval $[t, t + h]$ with probability $\lambda h + o(h)$, independently of the other cells, for some constant $\lambda > 0$. The number of cells at time t , $N(t)$, can thus be modeled as a continuous-time Markov chain. (a) Sketch an appropriate transition rate diagram and describe the transition rate matrix. (b) For fixed $t \geq 0$, let $P(z, t)$ denote the z -transform of the probability vector $\pi(t)$. Starting with the Kolmogorov forward equations for $(\pi(t))$, derive a partial differential equation for the z -transform of $\pi(t)$. Show that the solution is

$$P(z, t) = \frac{ze^{-\lambda t}}{1 - z + ze^{-\lambda t}}$$

(c) Find $E[N(t)]$. (d) Solve for $\pi(t)$. (e) Another population evolves deterministically. The first cell splits at time λ^{-1} , and thereafter each cell splits into two exactly λ^{-1} time units after its creation. Find the size of the population of this population as a function of time, and compare to your answer in part (c).

1.8. Equilibrium distribution of the jump chain

Suppose that π is the equilibrium distribution for a time-homogeneous Markov process with transition rate matrix Q . Suppose that $B^{-1} = \sum_i -q_{ii}\pi_i$, where the sum is over all i in the state space, is finite. Show that the equilibrium distribution for the jump chain $(X^J(k) : k \geq 0)$ is given by $\pi_i^J = -Bq_{ii}\pi_i$. (So π and π^J are identical if and only if q_{ii} is the same for all i .)

1.9. A simple Poisson process calculation

1. Let $(N(t) : t \geq 0)$ be a Poisson random process with rate $\lambda > 0$. Compute $P[N(s) = i | N(t) = k]$ where $0 < s < t$ and i and k are nonnegative integers. (Caution: note order of s and t carefully).

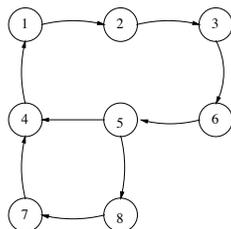
1.10. An alternating renewal process

Consider a two-color traffic signal on a street corner which alternately stays red for one minute and then stays green for one minute. Suppose you are riding your bicycle towards the signal and are a block away when you first spot it, so that it will take you one-half minute to reach the signal. (a) What is the probability that the signal will change colors (at least once) before you reach the corner? Explain. (b) Compute the conditional expectation of how long you must wait at the corner (possibly not at all) given that the light is green when you first spot it. Repeat assuming the light is red when you first spot it. (Which is greater? What is the average of the two answers?) (c) Repeat part (a) if instead the signal randomly changes colors, staying the same color for a time period uniformly distributed between 0 and 2 minutes. The times between switches are assumed to be independent. (d) How fast are you riding? (Hint: What units of speed can you use so that you don't need more information?)

1.10.5. A simple question of periods

3. (5 points) Consider a discrete-time Markov process with the nonzero one-step transition prob-

abilities indicated by the following graph.



- (a) What is the period of state 4?
- (b) What is the period of state 6?

1.11. A mean hitting time problem

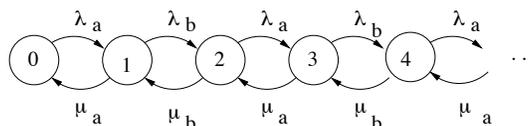
Let $(X(t) : t \geq 0)$ be a time-homogeneous, pure-jump Markov process with state space $\{0, 1, 2\}$ and Q matrix

$$Q = \begin{pmatrix} -4 & 2 & 2 \\ 1 & -2 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

- (a) Write down the state transition diagram and compute the equilibrium distribution.
- (b) Compute $a_i = E[\min\{t \geq 0 : X(t) = 1\} | X(0) = i]$ for $i = 0, 1, 2$. If possible, use an approach that can be applied to larger state spaces.
- (c) Derive a variation of the Kolmogorov forward differential equations for the quantities: $\alpha_i(t) = P[X(s) \neq 2 \text{ for } 0 \leq s \leq t \text{ and } X(t) = i | X(0) = 0]$ for $0 \leq i \leq 2$. (You need not solve the equations.)
- (d) The forward Kolmogorov equations describe the evolution of an initial probability distribution going forward in time, given an initial. In other problems, a boundary condition is given at a final time, and a differential equation working backwards in time from a final condition is called for (called Kolmogorov backward equations). Derive a backward differential equation for: $\beta_j(t) = P[X(s) \neq 2 \text{ for } t \leq s \leq t_f | X(t) = j]$, for $0 \leq j \leq 2$ and $t \leq t_f$ for some fixed time t_f . (Hint: Express $\beta_i(t-h)$ in terms of the $\beta_j(t)'s$ for $t \leq t_f$, and let $h \rightarrow 0$. You need not solve the equations.)

1.12. A birth-death process with periodic rates

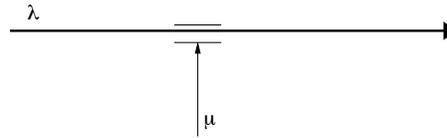
Consider a single server queueing system in which the number in the system is modeled as a continuous time birth-death process with the transition rate diagram shown, where $\lambda_a, \lambda_b, \mu_a,$ and μ_b are strictly positive constants.



- (a) Under what additional assumptions on these four parameters is the process positive recurrent?
- (b) Assuming the system is positive recurrent, under what conditions on $\lambda_a, \lambda_b, \mu_a,$ and μ_b is it true that the distribution of the number in the system at the time of a typical arrival is the same as the equilibrium distribution of the number in the system?

1.13. Markov model for a link with resets

Suppose that a regulated communication link resets at a sequence of times forming a Poisson process with rate μ . Packets are offered to the link according to a Poisson process with rate λ . Suppose the link shuts down after three packets pass in the absence of resets. Once the link is shut down, additional offered packets are dropped, until the link is reset again, at which time the process begins anew.



- (a) Sketch a transition rate diagram for a finite state Markov process describing the system state.
 (b) Express the dropping probability (same as the long term fraction of packets dropped) in terms of λ and μ .

1.14. An unusual birth-death process

Consider the birth-death process X with arrival rates $\lambda_k = (p/(1-p))^k/a_k$ and death rates $\mu_k = (p/(1-p))^{k-1}/a_k$, where $.5 < p < 1$, and $a = (a_0, a_1, \dots)$ is a probability distribution on the nonnegative integers with $a_k > 0$ for all k . (a) Classify the states for the process X as transient, null-recurrent or positive recurrent. (b) Check that $aQ = 0$. Is a an equilibrium distribution for X ? Explain. (c) Find the one-step transition probabilities for the jump-chain, X^J (d) Classify the states for the process X^J as transient, null-recurrent or positive recurrent.

1.15. A queue with decreasing service rate

Consider a queueing system in which the arrival process is a Poisson process with rate λ . Suppose the instantaneous completion rate is μ when there are K or fewer customers in the system, and $\mu/2$ when there are $K + 1$ or more customers in the system. The number in the system is modeled as a birth-death Markov process. (a) Sketch the transition rate diagram. (b) Under what condition on λ and μ are all states positive recurrent? Under this condition, give the equilibrium distribution. (c) Suppose that $\lambda = (2/3)\mu$. Describe in words the typical behavior of the system, given that it is initially empty.

1.16. Limit of a discrete time queueing system

Model a queue by a discrete-time Markov chain by recording the queue state after intervals of q seconds each. Assume the queue evolves during one of the atomic intervals as follows: There is an arrival during the interval with probability αq , and no arrival otherwise. If there is a customer in the queue at the beginning of the interval then a single departure will occur during the interval with probability βq . Otherwise no departure occurs. Suppose that it is impossible to have an arrival and a departure in a single atomic interval. (a) Find $a_k = P[\text{an interarrival time is } kq]$ and $b_k = P[\text{a service time is } kq]$. (b) Find the equilibrium distribution, $p = (p_k : k \geq 0)$, of the number of customers in the system at the end of an atomic interval. What happens as $q \rightarrow 0$?

1.17. An M/M/1 queue with impatient customers

Consider an M/M/1 queue with parameters λ and μ with the following modification. Each customer in the queue will defect (i.e. depart without service) with probability $\alpha h + o(h)$ in an interval of length h , independently of the other customers in the queue. Once a customer makes it to

the server it no longer has a chance to defect and simply waits until its service is completed and then departs from the system. Let $N(t)$ denote the number of customers in the system (queue plus server) at time t . (a) Give the transition rate diagram and generator matrix Q for the Markov chain $N = (N(t) : t \geq 0)$. (b) Under what conditions are all states positive recurrent? Under this condition, find the equilibrium distribution for N . (You need not explicitly sum the series.) (c) Suppose that $\alpha = \mu$. Find an explicit expression for p_D , the probability that a typical arriving customer defects instead of being served. Does your answer make sense as λ/μ converges to zero or to infinity?

1.18. Statistical multiplexing

Consider the following scenario regarding a one-way link in a store-and-forward packet communication network. Suppose that the link supports eight connections, each generating traffic at 5 kilobits per second (kbps). The data for each connection is assumed to be in packets exponentially distributed in length with mean packet size 1 kilobit. The packet lengths are assumed mutually independent and the packets for each stream arrive according to a Poisson process. Packets are queued at the beginning of the link if necessary, and queue space is unlimited. Compute the mean delay (queueing plus transmission time—neglect propagation delay) for each of the following three scenarios. Compare your answers. (a) (Full multiplexing) The link transmit speed is 50 kbps. (b) The link is replaced by two 25 kbps links, and each of the two links carries four sessions. (Of course the delay would be larger if the sessions were not evenly divided.) (c) (Multiplexing over two links) The link is replaced by two 25 kbps links. Each packet is transmitted on one link or the other, and neither link is idle whenever a packet from any session is waiting.

1.19. A queue with blocking

(M/M/1/5 system) Consider an M/M/1 queue with service rate μ , arrival rate λ , and the modification that at any time, at most five customers can be in the system (including the one in service, if any). If a customer arrives and the system is full (i.e. already has five customers in it) then the customer is dropped, and is said to be blocked. Let $N(t)$ denote the number of customers in the system at time t . Then $(N(t) : t \geq 0)$ is a Markov chain. (a) Indicate the transition rate diagram of the chain and find the equilibrium probability distribution. (b) What is the probability, p_B , that a typical customer is blocked? (c) What is the mean waiting time in queue, W , of a typical customer that is not blocked? (d) Give a simple method to numerically calculate, or give a simple expression for, the mean length of a busy period of the system. (A busy period begins with the arrival of a customer to an empty system and ends when the system is again empty.)

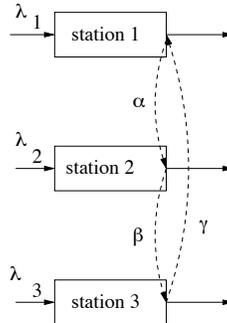
1.20. Multiplexing circuit and packet data streams

Consider a communication link that is shared between circuit switched traffic (consisting of “calls”) and datagram packet traffic. Suppose new calls arrive at rate λ_c , that new packets arrive at rate λ_p , and that the call durations are exponential with parameter δ_c . All calls are “served” simultaneously. Suppose that the link capacity is exactly sufficient to carry C calls, and that calls have priority over packets. Thus, a new call is blocked and lost if and only if it arrives to find C calls already in progress. Packets are not blocked, but instead they are queued until transmission. Finally, suppose that the instantaneous service rate of packets is $\mu(C - n_c)$ when there are n_c calls in progress. (a) Define a continuous-time, countable state Markov chain to model the system, and indicate the transition rates. (b) Is the method of Markov processes with phases relevant to this problem? If so, describe in detail how the method applies. (c) What is the necessary and sufficient condition

for the system to be stable (be as explicit as possible).

1.20.1. Three queues and an autonomously traveling server

Consider three stations that are served by a single rotating server, as pictured.



Customers arrive to station i according to a Poisson process of rate λ_i for $1 \leq i \leq 3$, and the total service requirement of each customer is exponentially distributed, with mean one. The rotation of the server is modelled by a three state Markov process with the transition rates α, β , and γ as indicated by the dashed lines. When at a station, the server works at unit rate, or is idle if the station is empty. If the service to a customer is interrupted because the server moves to the next station, the service is resumed when the server returns.

- Under what condition is the system stable? Briefly justify your answer.
- Identify a method for computing the mean customer waiting time at station one.

1.21. On two distributions seen by customers

Consider a queueing system in which the number in the system only changes in steps of plus one or minus one. Let $D(k, t)$ denote the number of customers that depart in the interval $[0, t]$ that leave behind exactly k customers, and let $R(k, t)$ denote the number of customers that arrive in the interval $[0, t]$ to find exactly k customers already in the system. (a) Show that $|D(k, t) - R(k, t)| \leq 1$ for all k and t . (b) Let α_t (respectively δ_t) denote the number of arrivals (departures) up to time t . Suppose that $\alpha_t \rightarrow \infty$ and $\alpha_t/\delta_t \rightarrow 1$ as $t \rightarrow \infty$. Show that if the following two limits exist for a given value k , then they are equal: $r_k = \lim_{t \rightarrow \infty} R(k, t)/\alpha_t$ and $d_k = \lim_{t \rightarrow \infty} D(k, t)/\delta_t$.

Chapter 2

Foster-Lyapunov stability criterion and moment bounds

Communication network models can become quite complex, especially when dynamic scheduling, congestion, and physical layer effects such as fading wireless channel models are included. It is thus useful to have methods to give approximations or bounds on key performance parameters. The criteria for stability and related moment bounds discussed in this chapter are useful for providing such bounds.

Aleksandr Mikhailovich Lyapunov (1857-1918) contributed significantly to the theory of stability of dynamical systems. Although a dynamical system may evolve on a complicated, multiple dimensional state space, a recurring theme of dynamical systems theory is that stability questions can often be settled by studying the potential of a system for some nonnegative potential function V . Potential functions used for stability analysis are widely called Lyapunov functions. Similar stability conditions have been developed by many authors for stochastic systems. Below we present the well known criteria due to Foster [14] for recurrence and positive recurrence. In addition we present associated bounds on the moments, which are expectations of some functions on the state space, computed with respect to the equilibrium probability distribution.¹

Section 2.1 discusses the discrete time tools, and presents examples involving load balancing routing, and input queued crossbar switches. Section 2.2 presents the continuous time tools, and an example. Problems are given in Section 2.3.

2.1 Stability criteria for discrete time processes

Consider an irreducible discrete-time Markov process X on a countable state space \mathcal{S} , with one-step transition probability matrix P . If f is a function on \mathcal{S} , then Pf represents the function obtained by multiplication of the vector f by the matrix P : $Pf(i) = \sum_{j \in \mathcal{S}} p_{ij} f(j)$. If f is nonnegative, then

¹The proof of Foster's criteria given here is similar to Foster's original proof, but is geared to establishing the moment bounds, and the continuous time versions, at the same time. The moment bounds and proofs given here are adaptations of those in Meyn and Tweedie [30] to discrete state spaces, and to continuous time. They involve some basic notions from martingale theory, which can be found, for example, in [18]. A special case of the moment bounds was given by Tweedie [42], and a version of the moment bound method was used by Kingman [26] in a queueing context. As noted in [30], the moment bound method is closely related to Dynkin's formula. The works [39, 40, 28, 41], and many others, have demonstrated the wide applicability of the stability methods in various queueing network contexts, using quadratic Lyapunov functions.

Pf is well defined, with the understanding that $Pf(i) = +\infty$ is possible for some, or all, values of i . An important property of Pf is that $Pf(i) = E[f(X(t+1))|X(t) = i]$. Let V be a nonnegative function on \mathcal{S} , to serve as the Lyapunov function. The drift function of $V(X(t))$ is defined by $d(i) = E[V(X(t+1))|X(t) = i] - V(i)$. That is, $d = PV - V$. Note that $d(i)$ is always well-defined, if the value $+\infty$ is permitted. The drift function is also given by

$$d(i) = \sum_{j:j \neq i} p_{ij}(V(j) - V(i)). \quad (2.1)$$

Proposition 2.1.1 (*Foster-Lyapunov stability criterion*) Suppose $V : \mathcal{S} \rightarrow \mathbb{R}_+$ and C is a finite subset of \mathcal{S} .

- (a) If $\{i : V(i) \leq K\}$ is finite for all K , and if $PV - V \leq 0$ on $\mathcal{S} - C$, then X is recurrent.
(b) If $\epsilon > 0$ and b is a constant such that $PV - V \leq -\epsilon + bI_C$, then X is positive recurrent.

The proof of the proposition is given after two lemmas.

Lemma 2.1.2 Suppose $PV - V \leq -f + g$ on \mathcal{S} , where f and g are nonnegative functions. Then for any initial state i_o and any stopping time τ ,

$$E \left[\sum_{k:0 \leq k \leq \tau-1} f(X(k)) \right] \leq V(i_o) + E \left[\sum_{k:0 \leq k \leq \tau-1} g(X(k)) \right] \quad (2.2)$$

Proof. Let \mathcal{F}_k denote the σ -algebra generated by $(X(s) : 0 \leq s \leq k)$. By assumption, $PV + f \leq V + g$ on \mathcal{S} . Evaluating each side of this inequality at $X(k)$, and taking the conditional expectation given \mathcal{F}_k yields

$$E[V(x(k+1))|\mathcal{F}_k] + f(X(k)) \leq V(X(k)) + g(X(k)). \quad (2.3)$$

Let $\tau^n = \min\{\tau, n, \inf\{k \geq 0 : V(X(k)) \geq n\}\}$. The next step is to multiply each side of (2.3) by $I_{\{\tau^n > k\}}$, take expectations of each side, and use the fact that

$$E [E[V(x(k+1))|\mathcal{F}_k]I_{\{\tau^n > k\}}] = E [V(x(k+1))I_{\{\tau^n > k\}}] \geq E [V(x(k+1))I_{\{\tau^n > k+1\}}].$$

The result is

$$E [V(x(k+1))I_{\{\tau^n > k+1\}}] + E [f(x(k))I_{\{\tau^n > k\}}] \leq E [V(x(k))I_{\{\tau^n > k\}}] + E [g(x(k))I_{\{\tau^n > k\}}] \quad (2.4)$$

The definition of τ^n implies that all the terms in (2.4) are zero for $k \geq n$, and that $E [V(x(k))I_{\{\tau^n > k\}}] < \infty$ for all k . Thus, it is legitimate to sum each side of (2.4) and cancel like terms, to yield:

$$E \left[\sum_{k:0 \leq k \leq \tau^n-1} f(X(k)) \right] \leq V(i_o) + E \left[\sum_{k:0 \leq k \leq \tau^n-1} g(X(k)) \right]. \quad (2.5)$$

Letting $n \rightarrow \infty$ in (2.5) and appealing to the monotone convergence theorem yields (2.2). ■

Lemma 2.1.3 Let X be an irreducible, time-homogeneous Markov process. If there is a finite set C , and the mean time to hit C starting from any state in C is finite, then X is positive recurrent.

Proof. Lemma I.3.9 of Assussen. Variation of Wald's lemma. ■

Proof of Proposition 2.1.1. (a) The proof of part (a) is based on a martingale convergence theorem. Fix an initial state $i_o \in \mathcal{S} - C$, let $\tau = \min\{t \geq 0 : X(t) \in C\}$, and let $Y(t) = V(X(t \wedge \tau))$ for $t \geq 0$. Note that $Y(t+1) - Y(t) = (V(X(t+1)) - V(X(t)))I_{\{\tau > t\}}$. Let \mathcal{F}_t denote the σ -algebra generated by $(X(s) : 0 \leq s \leq t)$, and note that the event $\{\tau > t\}$ is in \mathcal{F}_t . Thus,

$$E[Y(t+1) - Y(t) | \mathcal{F}_t] = E[(V(X(t+1)) - V(X(t)))I_{\{\tau > t\}} | \mathcal{F}_t] = E[V(X(t+1)) - V(X(t)) | \mathcal{F}_t] I_{\{\tau > t\}} \geq 0,$$

so that $(Y_t : t \geq 0)$ is a nonnegative supermartingale. Therefore, by a version of the martingale convergence theorem, $\lim_{t \rightarrow \infty} Y(t)$ exists and is finite with probability one. By the assumption $\{i : V(i) \leq K\}$ is finite for all K , it follows that $X(t \wedge \tau)$ reaches only finitely many states with probability one, which implies that $\tau < +\infty$ with probability one. Therefore, $P[\tau < \infty | X(0) = i_o] = 1$ for any $i_o \in \mathcal{S} - C$. Therefore, for any initial state in C , the process returns to C infinitely often with probability one. The process watched in C is a finite-state Markov process, and is thus (positive) recurrent. Therefore, all the states of C are recurrent for the original process X . Part (a) is proved.

(b) Let $f \equiv -\epsilon$, $g = bI_C$, and $\tau = \min\{t \geq 1 : X(t) \in C\}$. Then Lemma 2.1.2 implies that $\epsilon E[\tau] \leq V(i_o) + b$ for any $i_o \in \mathcal{S}$. In particular, the mean time to hit C after time zero, beginning from any state in C , is finite. Therefore X is positive recurrent by Lemma 2.1.3. ■

Proposition 2.1.4 (*Moment bound*) Suppose V , f , and g are nonnegative functions on \mathcal{S} and suppose

$$PV(i) - V(i) \leq -f(i) + g(i) \quad \text{for all } i \in \mathcal{S} \quad (2.6)$$

In addition, suppose X is positive recurrent, so that the means, $\bar{f} = \pi f$ and $\bar{g} = \pi g$ are well-defined. Then $\bar{f} \leq \bar{g}$. (In particular, if g is bounded, then \bar{g} is finite, and therefore \bar{f} is finite.)

Proof. Fix a state i_o and let T_m be the time of the m^{th} return to state i_o . Then by the equality of time and statistical averages,

$$E \left[\sum_{\{k: 0 \leq k \leq T_m - 1\}} f(X(k)) \right] = mE[T_1] \bar{f}$$

$$E \left[\sum_{\{k: 0 \leq k \leq T_m - 1\}} g(X(k)) \right] = mE[T_1] \bar{g}$$

Lemma 2.1.2 applied with stopping time T_m thus yields $mE[T_1] \bar{f} \leq V(i_o) + mE[T_1] \bar{g}$. Dividing through by $mE[T_1]$ and letting $m \rightarrow \infty$ yields the desired inequality, $\bar{f} \leq \bar{g}$. ■

Corollary 2.1.5 (*Combined Foster-Lyapunov stability criterion and moment bound*) Suppose V , f , and g are nonnegative functions on \mathcal{S} such that

$$PV(i) - V(i) \leq -f(i) + g(i) \quad \text{for all } i \in \mathcal{S} \quad (2.7)$$

In addition, suppose for some $\epsilon > 0$ that the set C defined by $C = \{i : f(i) < g(i) + \epsilon\}$ is finite. Then X is positive recurrent and $\bar{f} \leq \bar{g}$. (In particular, if g is bounded, then \bar{g} is finite, and therefore \bar{f} is finite.)

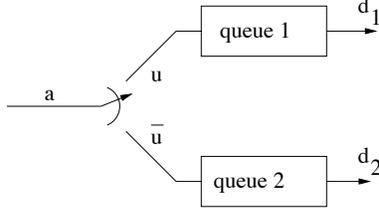


Figure 2.1: Two queues fed by a single arrival stream.

Proof. Let $b = \max\{g(i) + \epsilon - f(i) : i \in C\}$. Then V, C, b , and ϵ satisfy the hypotheses of Proposition 2.1.1(b), so that X is positive recurrent. Therefore the hypotheses of Proposition 2.1.4 are satisfied, so that $\bar{f} \leq \bar{g}$. ■

The assumptions in Propositions 2.1.1 and 2.1.4 and Corollary 2.1.5 do not imply that \bar{V} is finite. Even so, since V is nonnegative, for a given initial state $X(0)$, the long term average drift of $V(X(t))$ is nonnegative. This gives an intuitive reason why the mean downward part of the drift, \bar{f} , must be less than or equal to the mean upward part of the drift, \bar{g} .

Example 1a (Probabilistic routing to two queues) Consider the routing scenario with two queues, queue 1 and queue 2, fed by a single stream of packets, as pictured in Figure 2.1. Here, $0 \leq a, u, d_1, d_2 \leq 1$, and $\bar{u} = 1 - u$. The state space for the process is $\mathcal{S} = \mathbb{Z}_+^2$, where the state $x = (x_1, x_2)$ denotes x_1 packets in queue 1 and x_2 packets in queue 2. In each time slot, a new arrival is generated with probability a , and then is routed to queue 1 with probability u and to queue 2 with probability \bar{u} . Then each queue i , if not empty, has a departure with probability d_i . Note that we allow a packet to arrive and depart in the same slot. Thus, if $X_i(t)$ is the number of packets in queue i at the beginning of slot t , then the system dynamics can be described as follows:

$$X_i(t+1) = X_i(t) + A_i(t) - D_i(t) + L_i(t) \quad \text{for } i \in \{0, 1\} \quad (2.8)$$

where

- $A(t) = (A_1(t), A_2(t))$ is equal to $(1, 0)$ with probability au , $(0, 1)$ with probability $a\bar{u}$, and $A(t) = (0, 0)$ otherwise.
- $D_i(t) : t \geq 0$, are *Bernoulli*(d_i) random variables, for $i \in \{0, 1\}$
- All the $A(t)$'s, $D_1(t)$'s, and $D_2(t)$'s are mutually independent
- $L_i(t) = -(X_i(t) + A_i(t) - D_i(t))_+$ (see explanation next)

If $X_i(t) + A_i(t) = 0$, there can be no actual departure from queue i . However, we still allow $D_i(t)$ to equal one. To keep the queue length process from going negative, we add the random variable $L_i(t)$ in (2.8). Thus, $D_i(t)$ is the *potential* number of departures from queue i during the slot, and $D_i(t) - L_i(t)$ is the actual number of departures. This completes the specification of the one-step transition probabilities of the Markov process.

A necessary condition for positive recurrence is, for any routing policy, $a < d_1 + d_2$, because the total arrival rate must be less than the total departure rate. We seek to show that this necessary condition is also sufficient, under the random routing policy.

Let us calculate the drift of $V(X(t))$ for the choice $V(x) = (x_1^2 + x_2^2)/2$. Note that $(X_i(t+1))^2 = (X_i(t) + A_i(t) - D_i(t) + L_i(t))^2 \leq (X_i(t) + A_i(t) - D_i(t))^2$, because addition of the variable $L_i(t)$ can only push $X_i(t) + A_i(t) - D_i(t)$ closer to zero. Thus,

$$\begin{aligned}
PV(x) - V(x) &= E[V(X(t+1))|X(t) = x] - V(x) \\
&\leq \frac{1}{2} \sum_{i=1}^2 E[(x_i + A_i(t) - D_i(t))^2 - x_i^2 | X(t) = x] \\
&= \sum_{i=1}^2 x_i E[A_i(t) - D_i(t) | X(t) = x] + \frac{1}{2} E[(A_i(t) - D_i(t))^2 | X(t) = x] \quad (2.9) \\
&\leq \left(\sum_{i=1}^2 x_i E[A_i(t) - D_i(t) | X(t) = x] \right) + 1 \\
&= -(x_1(d_1 - au) + x_2(d_2 - a\bar{u})) + 1 \quad (2.10)
\end{aligned}$$

Under the necessary condition $a < d_1 + d_2$, there are choices of u so that $au < d_1$ and $a\bar{u} < d_2$, and for such u the conditions of Corollary 2.1.5 are satisfied, with $f(x) = x_1(d_1 - au) + x_2(d_2 - a\bar{u})$, $g(x) = 1$, and any $\epsilon > 0$, implying that the Markov process is positive recurrent. In addition, the first moments under the equilibrium distribution satisfy:

$$(d_1 - au)\bar{X}_1 + (d_2 - a\bar{u})\bar{X}_2 \leq 1. \quad (2.11)$$

In order to deduce an upper bound on $\bar{X}_1 + \bar{X}_2$, we select u^* to maximize the minimum of the two coefficients in (2.11). Intuitively, this entails selecting u to minimize the absolute value of the difference between the two coefficients. We find:

$$\begin{aligned}
\epsilon &= \max_{0 \leq u \leq 1} \min\{d_1 - au, d_2 - a\bar{u}\} \\
&= \min\left\{d_1, d_2, \frac{d_1 + d_2 - a}{2}\right\}
\end{aligned}$$

and the corresponding value u^* of u is given by

$$u^* = \begin{cases} 0 & \text{if } d_1 - d_2 < -a \\ \frac{1}{2} + \frac{d_1 - d_2}{2a} & \text{if } |d_1 - d_2| \leq a \\ 1 & \text{if } d_1 - d_2 > a \end{cases}$$

For the system with $u = u^*$, (2.11) yields

$$\bar{X}_1 + \bar{X}_2 \leq \frac{1}{\epsilon}. \quad (2.12)$$

We remark that, in fact,

$$\bar{X}_1 + \bar{X}_2 \leq \frac{2}{d_1 + d_2 - a} \quad (2.13)$$

If $|d_1 - d_2| \leq a$ then the bounds (2.12) and (2.13) coincide, and otherwise, the bound (2.13) is strictly tighter. If $d_1 - d_2 < -a$ then $u^* = 0$, so that $\bar{X}_1 = 0$, and (2.11) becomes $(d_2 - a)\bar{X}_2 \leq 1$, which implies (2.13). Similarly, if $d_1 - d_2 > a$, then $u^* = 1$, so that $\bar{X}_2 = 0$, and (2.11) becomes

$(d_1 - a)\overline{X}_1 \leq 1$, which implies (2.13). Thus, (2.13) is proved.

Example 1b (Route-to-shorter policy) Consider a variation of the previous example such that when a packet arrives, it is routed to the shorter queue. To be definite, in case of a tie, the packet is routed to queue 1. Then the evolution equation (2.8) still holds, but with with the description of the arrival variables changed to the following:

- Given $X(t) = (x_1, x_2)$, $A(t) = (I_{\{x_1 \leq x_2\}}, I_{\{x_1 > x_2\}})$ with probability a , and $A(t) = (0, 0)$ otherwise. Let P^{RS} denote the one-step transition probability matrix when the route-to-shorter policy is used.

Proceeding as in (2.9) yields:

$$\begin{aligned} P^{RS}V(x) - V(x) &\leq \sum_{i=1}^2 x_i E[A_i(t) - D_i(t) | X(t) = x] + 1 \\ &= a(x_1 I_{\{x_1 \leq x_2\}} + x_2 I_{\{x_1 > x_2\}}) - d_1 x_1 - d_2 x_2 + 1 \end{aligned}$$

Note that $x_1 I_{\{x_1 \leq x_2\}} + x_2 I_{\{x_1 > x_2\}} \leq ux_1 + \bar{u}x_2$ for any $u \in [0, 1]$, with equality for $u = I_{\{x_1 \leq x_2\}}$. Therefore, the drift bound for V under the route-to-shorter policy is less than or equal to the drift bound (2.10), for V for any choice of probabilistic splitting. In fact, route-to-shorter routing can be viewed as a controlled version of the independent splitting model, for which the control policy is selected to minimize the bound on the drift of V in each state. It follows that the route-to-shorter process is positive recurrent as long as $a < d_1 + d_2$, and (2.11) holds for any value of u such that $au < d_1$ and $a\bar{u} \leq d_2$. In particular, (2.12) holds for the route-to-shorter process.

We remark that the stronger bound (2.13) is not always true for the route-to-shorter policy. The problem is that even if $d_1 - d_2 < -a$, the route-to-shorter policy can still route to queue 1, and so $\overline{X}_1 \neq 0$. In fact, if a and d_2 are fixed with $0 < a < d_2 < 1$, then $\overline{X}_1 \rightarrow \infty$ as $d_1 \rightarrow 0$ for the route-to-shorter policy. Intuitively, that is because occasionally there will be a large number of customers in the system due to statistical fluctuations, and then there will be many customers in queue 1. But if $d_2 \ll 1$, those customers will remain in queue 2 for a very long time.

Example 2a (An input queued switch with probabilistic switching)² Consider a packet switch with N inputs and N outputs, as pictured in Figure 2.2. Suppose there are N^2 queues – N at each input – with queue i, j containing packets that arrived at input i and are destined for output j , for $i, j \in E$, where $E = \{1, \dots, N\}$. Suppose the packets are all the same length, and adopt a discrete time model, so that during one time slot, a transfer of packets can occur, such that at most one packet can be transferred from each input, and at most one packet can be transferred to each output. A permutation σ of E has the form $\sigma = (\sigma_1, \dots, \sigma_N)$, where $\sigma_1, \dots, \sigma_N$ are distinct elements of E . Let Π denote the set of all $N!$ such permutations. Given $\sigma \in \Pi$, let $R(\sigma)$ be the $N \times N$ *switching matrix* defined by $R_{ij} = I_{\{\sigma_i=j\}}$. Thus, $R_{ij}(\sigma) = 1$ means that under permutation σ , input i is connected to output j , or, equivalently, a packet in queue i, j is to depart, if there is any such packet. A state x of the system has the form $x = (x_{ij} : i, j \in E)$, where x_{ij} denotes the number of packets in queue i, j .

²Tassioulas [41] originally developed the results of Examples 2a-b, in the context of wireless networks. The paper [29] presents similar results in the context of a packet switch.

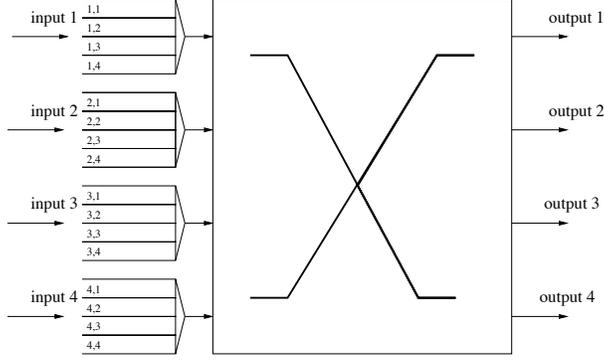


Figure 2.2: A 4×4 input queued switch

The evolution of the system over a time slot $[t, t + 1)$ is described as follows:

$$X_{ij}(t + 1) = X_{ij}(t) + A_{ij}(t) - R_{ij}(\sigma(t)) + L_{ij}(t)$$

where

- $A_{ij}(t)$ is the number of packets arriving at input i , destined for output j , in the slot. Assume that the variables $(A_{ij}(t) : i, j \in E, t \geq 0)$ are mutually independent, and for each i, j , the random variables $(A_{ij}(t) : t \geq 0)$ are independent, identically distributed, with mean λ_{ij} and $E[A_{ij}^2] \leq K_{ij}$, for some constants λ_{ij} and K_{ij} . Let $\Lambda = (\lambda_{ij} : i, j \in E)$.
- $\sigma(t)$ is the switch state used during the slot
- $L_{ij} = -(X_{ij}(t) + A_{ij}(t) - R_{ij}(\sigma(t)))_+$, which takes value one if there was an unused potential departure at queue ij during the slot, and is zero otherwise.

The number of packets at input i at the beginning of the slot is given by the row sum $\sum_{j \in E} X_{ij}(t)$, its mean is given by the row sum $\sum_{j \in E} \lambda_{ij}$, and at most one packet at input i can be served in a time slot. Similarly, the set of packets waiting for output j , called the *virtual queue* for output j , has size given by the column sum $\sum_{i \in E} X_{ij}(t)$. The mean number of arrivals to the virtual queue for output j is $\sum_{i \in E} \lambda_{ij}(t)$, and at most one packet in the virtual queue can be served in a time slot. These considerations lead us to impose the following restrictions on Λ :

$$\sum_{j \in E} \lambda_{ij} < 1 \text{ for all } i \quad \text{and} \quad \sum_{i \in E} \lambda_{ij} < 1 \text{ for all } j \quad (2.14)$$

Except for trivial cases involving deterministic arrival sequences, the conditions (2.14) are necessary for stable operation, for any choice of the switch schedule $(\sigma(t) : t \geq 0)$.

Let's first explore random, independent and identically distributed (i.i.d.) switching. That is, given a probability distribution u on Π , let $(\sigma(t) : t \geq 0)$ be independent with common probability distribution u . Once the distributions of the A_{ij} 's and u are fixed, we have a discrete-time Markov process model. Given Λ satisfying (2.14), we wish to determine a choice of u so that the process with i.i.d. switch selection is positive recurrent.

Some standard background from switching theory is given in this paragraph. A *line sum* of a matrix M is either a row sum, $\sum_j M_{ij}$, or a column sum, $\sum_i M_{ij}$. A square matrix M is called

doubly stochastic if it has nonnegative entries and if all of its line sums are one. Birkhoff's theorem, celebrated in the theory of switching, states that any doubly stochastic matrix M is a convex combination of switching matrices. That is, such an M can be represented as $M = \sum_{\sigma \in \Pi} R(\sigma)u(\sigma)$, where $u = (u(\sigma) : \sigma \in \Pi)$ is a probability distribution on Π . If \widetilde{M} is a nonnegative matrix with all line sums less than or equal to one, then if some of the entries of \widetilde{M} are increased appropriately, a doubly stochastic matrix can be obtained. That is, there exists a doubly stochastic matrix M so that $\widetilde{M}_{ij} \leq M_{ij}$ for all i, j . Applying Birkhoff's theorem to M yields that there is a probability distribution u so that $\widetilde{M}_{ij} \leq \sum_{\sigma \in \Pi} R(\sigma)u(\sigma)$ for all i, j .

Suppose Λ satisfies the necessary conditions (2.14). That is, suppose that all the line sums of Λ are less than one. Then with ϵ defined by

$$\epsilon = \frac{1 - (\text{maximum line sum of } \Lambda)}{N},$$

each line sum of $(\lambda_{ij} + \epsilon : i, j \in E)$ is less than or equal to one. Thus, by the observation at the end of the previous paragraph, there is a probability distribution u^* on Π so that $\lambda_{ij} + \epsilon \leq \mu_{ij}(u^*)$, where

$$\mu_{ij}(u) = \sum_{\sigma \in \Pi} R_{ij}(\sigma)u(\sigma).$$

We consider the system using probability distribution u^* for the switch states. That is, let $(\sigma(t) : t \geq 0)$ be independent, each with distribution u^* . Then for each ij , the random variables $R_{ij}(\sigma(t))$ are independent, Bernoulli($\mu_{ij}(u^*)$) random variables.

Consider the quadratic Lyapunov function V given by $V(x) = \frac{1}{2} \sum_{i,j} x_{ij}^2$. As in (2.9),

$$PV(x) - V(x) \leq \sum_{i,j} x_{ij} E[A_{ij}(t) - R_{ij}(\sigma(t)) | X_{ij}(t) = x] + \frac{1}{2} \sum_{i,j} E[(A_{ij}(t) - R_{ij}(\sigma(t)))^2 | X(t) = x].$$

Now

$$E[A_{ij}(t) - R_{ij}(\sigma(t)) | X_{ij}(t) = x] = E[A_{ij}(t) - R_{ij}(\sigma(t))] = \lambda_{ij} - \mu_{ij}(u^*) \leq -\epsilon$$

and

$$\frac{1}{2} \sum_{i,j} E[(A_{ij}(t) - R_{ij}(\sigma(t)))^2 | X(t) = x] \leq \frac{1}{2} \sum_{i,j} E[(A_{ij}(t))^2 + (R_{ij}(\sigma(t)))^2] \leq K$$

where $K = \frac{1}{2}(N + \sum_{i,j} K_{ij})$. Thus,

$$PV(x) - V(x) \leq -\epsilon \left(\sum_{ij} x_{ij} \right) + K \quad (2.15)$$

Therefore, by Corollary 2.1.5, the process is positive recurrent, and

$$\sum_{ij} \bar{X}_{ij} \leq \frac{K}{\epsilon} \quad (2.16)$$

That is, the necessary condition (2.14) is also sufficient for positive recurrence and finite mean queue length in equilibrium, under i.i.d. random switching, for an appropriate probability distribution u^* on the set of permutations.

Example 2b (An input queued switch with maximum weight switching) The random switching policy used in Example 2a depends on the arrival rate matrix Λ , which may be unknown a priori. Also, the policy allocates potential departures to a given queue ij , whether or not the queue is empty, even if other queues could be served instead. This suggests using a dynamic switching policy, such as the *maximum weight* switching policy, defined by $\sigma(t) = \sigma^{MW}(X(t))$, where for a state x ,

$$\sigma^{MW}(x) = \arg \max_{\sigma \in \Pi} \sum_{ij} x_{ij} R_{ij}(\sigma). \quad (2.17)$$

The use of “arg max” here means that $\sigma^{MW}(x)$ is selected to be a value of σ that maximizes the sum on the right hand side of (2.17), which is the weight of permutation σ with edge weights x_{ij} . In order to obtain a particular Markov model, we assume that the set of permutations Π is numbered from 1 to $N!$ in some fashion, and in case there is a tie between two or more permutations for having the maximum weight, the lowest numbered permutation is used. Let P^{MW} denote the one-step transition probability matrix when the route-to-shorter policy is used.

Letting V and K be as in Example 2a, we find under the maximum weight policy that

$$P^{MW}V(x) - V(x) \leq \sum_{ij} x_{ij}(\lambda_{ij} - R_{ij}(\sigma^{MW}(x))) + K$$

The maximum of a function is greater than or equal to the average of the function, so that for any probability distribution u on Π

$$\begin{aligned} \sum_{ij} x_{ij} R_{ij}(\sigma^{MW}(t)) &\geq \sum_{\sigma} u(\sigma) \sum_{ij} x_{ij} R_{ij}(\sigma) \\ &= \sum_{ij} x_{ij} \mu_{ij}(u) \end{aligned} \quad (2.18)$$

with equality in (2.18) if and only if u is concentrated on the set of maximum weight permutations. In particular, the choice $u = u^*$ shows that

$$\sum_{ij} x_{ij} R_{ij}(\sigma^{MW}(t)) \geq \sum_{ij} x_{ij} \mu_{ij}(u^*) \geq \sum_{ij} x_{ij}(\lambda_{ij} + \epsilon)$$

Therefore, if P is replaced by P^{MW} , (2.15) still holds. Therefore, by Corollary 2.1.5, the process is positive recurrent, and the same moment bound, (2.16), holds, as for the randomized switching strategy of Example 2a. On one hand, implementing the maximum weight algorithm does not require knowledge of the arrival rates, but on the other hand, it requires that queue length information be shared, and that a maximization problem be solved for each time slot. Much recent work has gone towards reduced complexity dynamic switching algorithms.

2.2 Stability criteria for continuous time processes

Here is a continuous time version of the Foster-Lyapunov stability criteria and the moment bounds. Suppose X is a time-homogeneous, irreducible, continuous-time Markov process with generator matrix Q . The drift vector of $V(X(t))$ is the vector QV . This definition is motivated by the fact

that the mean drift of X for an interval of duration h is given by

$$\begin{aligned}
d_h(i) &= \frac{E[V(X(t+h))|X(t)=i] - V(i)}{h} \\
&= \sum_{j \in \mathcal{S}} \left(\frac{p_{ij}(h) - \delta_{ij}}{h} \right) V(j) \\
&= \sum_{j \in \mathcal{S}} \left(q_{ij} + \frac{o(h)}{h} \right) V(j),
\end{aligned} \tag{2.19}$$

so that if the limit as $h \rightarrow 0$ can be taken inside the summation in (2.19), then $d_h(i) \rightarrow QV(i)$ as $h \rightarrow 0$. The following useful expression for QV follows from the fact that the row sums of Q are zero:

$$QV(i) = \sum_{j:j \neq i} q_{ij}(V(j) - V(i)). \tag{2.20}$$

Formula (2.20) is quite similar to the formula (2.1) for the drift vector for a discrete-time process.

Proposition 2.2.1 (*Foster-Lyapunov stability criterion—continuous time*) Suppose $V : \mathcal{S} \rightarrow \mathbb{R}_+$ and C is a finite subset of \mathcal{S} .

- (a) If $QV \leq 0$ on $\mathcal{S} - C$, and $\{i : V(i) \leq K\}$ is finite for all K then X is recurrent.
(b) Suppose for some $b > 0$ and $\epsilon > 0$ that

$$QV(i) \leq -\epsilon + bI_C(i) \quad \text{for all } i \in \mathcal{S}. \tag{2.21}$$

Suppose further that $\{i : V(i) \leq K\}$ is finite for all K , or that X is nonexplosive. Then X is positive recurrent.

The proof of the proposition is similar to that above for discrete time processes. We begin with an analog of Lemma 2.1.2.

Lemma 2.2.2 Suppose $QV \leq -f + g$ on \mathcal{S} , where f and g are nonnegative functions. Fix an initial state i_o , let N be a stopping time for the jump process X^J , and let τ^N denote the time of the N^{th} jump of X . Then

$$E \left[\int_0^{\tau^N} f(X(t)) dt \right] \leq V(i_o) + E \left[\int_0^{\tau^N} g(X(t)) dt \right] \tag{2.22}$$

Proof. Let D denote the diagonal matrix with entries $-q_{ii}$. The one-step transition probability matrix of the jump chain is given by $P^J = D^{-1}Q + I$. The condition $QV \leq -f + g$ thus implies $P^J V - V \leq -\tilde{f} + \tilde{g}$, where $\tilde{f} = D^{-1}f$ and $\tilde{g} = D^{-1}g$. Lemma 2.1.2 applied to the jump chain thus yields

$$E \left[\sum_{k:0 \leq k \leq \tau-1} \tilde{f}(X^J(k)) \right] \leq V(i_o) + E \left[\sum_{k:0 \leq k \leq \tau-1} \tilde{g}(X^J(k)) \right] \tag{2.23}$$

However, by the space-time description of an excursion of X from i_o ,

$$E \left[\sum_{k:0 \leq k \leq N-1} \tilde{f}(X^J(k)) \right] = E \left[\int_0^{\tau^N} f(X(t)) dt \right]$$

and a similar equation holds for \tilde{g} and g . Substituting these relations into (2.23) yields (2.22), as desired. \blacksquare

Proof of Proposition 2.2.1. (a) Let D denote the diagonal matrix with entries $-q_{ii}$. The one-step transition probability matrix of the jump chain is given by $P^J = D^{-1}Q + I$. The condition $QV \leq 0$ thus implies $P^J V - V \leq 0$, so that X^J is recurrent by Proposition 2.1.1(a). But X^J is recurrent if and only if X is recurrent, so X is also recurrent.

(b) The assumptions imply $QV \leq 0$ on $\mathcal{S} - C$, so if $\{i : V(i) \leq K\}$ is finite for all K then X is recurrent by Proposition 2.2.1(a), and, in particular, X is nonexplosive. Thus, X is nonexplosive, either by direct assumption or by implication.

Let $f \equiv -\epsilon$, let $g = bI_C$, let $i_o \in C$, let $N = \min\{k \geq 1 : X^J(k) \in C\}$, and let τ_N denote the time of the N^{th} jump. Then Lemma 2.2.2 implies that $\epsilon E[\tau^N] \leq V(i_o) + b/q_{i_o i_o}$. Since τ_N is finite a.s. and since X is not explosive, it must be that τ_N is the time that X returns to the set C after exiting the initial state i_o . That is, the time to hit C beginning from any state in C is finite. Therefore X is positive recurrent by the continuous time version of Lemma 2.1.3. \blacksquare

Example 3 Suppose X has state space $\mathcal{S} = \mathbb{Z}_+$, with $q_{i0} = \mu$ for all $i \geq 1$, $q_{ii+1} = \lambda_i$ for all $i \geq 0$, and all other off-diagonal entries of the rate matrix Q equal to zero, where $\mu > 0$ and $\lambda_i > 0$ such that $\sum_{i \geq 0} \frac{1}{\lambda_i} < +\infty$. Let $C = \{0\}$, $V(0) = 0$, and $V(i) = 1$ for $i \geq 1$. Then $QV = -\mu + (\lambda_0 + \mu)I_C$, so that (2.21) is satisfied with $\epsilon = \mu$ and $b = \lambda_0 + \mu$. However, X is not positive recurrent. In fact, X is explosive. To see this, note that $p_{ii+1}^J = \frac{\lambda_i}{\mu + \lambda_i} \geq \exp(-\frac{\mu}{\lambda_i})$. Let δ be the probability that, starting from state 0, the jump process does not return to zero. Then $\delta = \prod_{i=0}^{\infty} p_{ii+1}^J \geq \exp(-\mu \sum_{i=0}^{\infty} \frac{1}{\lambda_i}) > 0$. Thus, X^J is transient. After the last visit to state zero, all the jumps of X^J are up one. The corresponding mean holding times of X are $\frac{1}{\lambda_i + \mu}$ which have a finite sum, so that the process X is explosive. This example illustrates the need for the assumption just after (2.21) in Proposition 2.2.1.

As for the case of discrete time, the drift conditions imply moment bounds. The proofs of the following two propositions are minor variations of the ones used for discrete time, with Lemma 2.2.2 used in place of Lemma 2.1.2, and are omitted.

Proposition 2.2.3 (*Moment bound–continuous time*) Suppose V , f , and g are nonnegative functions on \mathcal{S} , and suppose $QV(i) \leq -f(i) + g(i)$ for all $i \in \mathcal{S}$. In addition, suppose X is positive recurrent, so that the means, $\bar{f} = \pi f$ and $\bar{g} = \pi g$ are well-defined. Then $\bar{f} \leq \bar{g}$.

Corollary 2.2.4 (*Combined Foster-Lyapunov stability criterion and moment bound–continuous time*) Suppose V , f , and g are nonnegative functions on \mathcal{S} such that $QV(i) \leq -f(i) + g(i)$ for all $i \in \mathcal{S}$, and, for some $\epsilon > 0$, the set C defined by $C = \{i : f(i) < g(i) + \epsilon\}$ is finite. Suppose also that $\{i : V(i) \leq K\}$ is finite for all K . Then X is positive recurrent and $\bar{f} \leq \bar{g}$.

Example 4.a (Random server allocation with two servers) Consider the system shown in Figure 2.3. Suppose that each queue i is fed by a Poisson arrival process with rate λ_i , and suppose there are two potential departure processes, D_1 and D_2 , which are Poisson processes with rates m_1 and m_2 , respectively. The five Poisson processes are assumed to be independent. No matter how the potential departures are allocated to the permitted queues, the following conditions are necessary for stability:

$$\lambda_1 < m_1, \quad \lambda_3 < m_2, \quad \text{and} \quad \lambda_1 + \lambda_2 + \lambda_3 < m_1 + m_2 \quad (2.24)$$

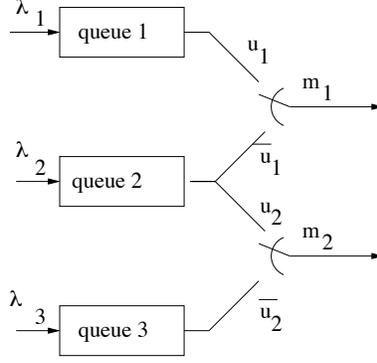


Figure 2.3: A system of three queues with two servers

That is because server 1 is the only one that can serve queue 1, server 2 is the only one that can serve queue 3, and the sum of the potential service rates must exceed the sum of the potential arrival rates for stability. A vector $x = (x_1, x_2, x_3) \in \mathbb{Z}_+^3$ corresponds to x_i packets in queue i for each i . Let us consider random selection, so that when D_i has a jump, the queue served is chosen at random, with the probabilities determined by $u = (u_1, u_2)$. As indicated in Figure 2.3, a potential service by server 1 is given to queue 1 with probability u_1 , and to queue 2 with probability \bar{u}_1 . Similarly, a potential service by server 2 is given to queue 2 with probability u_2 , and to queue 3 with probability \bar{u}_2 . The rates of potential service at the three stations are given by

$$\begin{aligned}\mu_1(u) &= u_1 m_1 \\ \mu_2(u) &= \bar{u}_1 m_1 + u_2 m_2 \\ \mu_3(u) &= \bar{u}_2 m_2.\end{aligned}$$

Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$. Using (2.20), we find that the drift function QV is given by

$$QV(x) = \frac{1}{2} \left(\sum_{i=1}^3 ((x_i + 1)^2 - x_i^2) \lambda_i \right) + \frac{1}{2} \left(\sum_{i=1}^3 ((x_i - 1)_+^2 - x_i^2) \mu_i(u) \right)$$

Now $(x_i - 1)_+^2 \leq (x_i - 1)^2$, so that

$$QV(x) \leq \left(\sum_{i=1}^3 x_i (\lambda_i - \mu_i(u)) \right) + \frac{\gamma}{2} \quad (2.25)$$

where γ is the total rate of events, given by $\gamma = \lambda_1 + \lambda_2 + \lambda_3 + \mu_1(u) + \mu_2(u) + \mu_3(u)$, or equivalently, $\gamma = \lambda_1 + \lambda_2 + \lambda_3 + m_1 + m_2$. Suppose that the necessary condition (2.24) holds. Then there exists some $\epsilon > 0$ and choice of u so that

$$\lambda_i + \epsilon \leq \mu_i(u) \quad \text{for } 1 \leq i \leq 3$$

and the largest such choice of ϵ is $\epsilon = \min\{m_1 - \lambda_1, m_2 - \lambda_3, \frac{m_1 + m_2 - \lambda_1 - \lambda_2 - \lambda_3}{3}\}$. (See exercise.) So $QV(x) \leq -\epsilon(x_1 + x_2 + x_3) + \gamma$ for all x , so Corollary 2.2.4 implies that X is positive recurrent and $\bar{X}_1 + \bar{X}_2 + \bar{X}_3 \leq \frac{\gamma}{2\epsilon}$.

Example 4.b (Longer first server allocation with two servers) This is a continuation of Example 4.a, concerned with the system shown in Figure 2.3. Examine the right hand side of (2.25). Rather than taking a fixed value of u , suppose that the choice of u could be specified as a function of the state x . The maximum of a function is greater than or equal to the average of the function, so that for any probability distribution u ,

$$\begin{aligned} \sum_{i=1}^3 x_i \mu_i(u) &\leq \max_{u'} \sum_i x_i \mu_i(u') \\ &= \max_{u'} m_1(x_1 u'_1 + x_2 \overline{u'_1}) + m_2(x_2 u'_2 + x_3 \overline{u'_2}) \\ &= m_1(x_1 \vee x_2) + m_2(x_2 \vee x_3) \end{aligned} \quad (2.26)$$

with equality in (2.26) for a given state x if and only if a longer first policy is used: each service opportunity is allocated to the longer queue connected to the server. Let Q^{LF} denote the one-step transition probability matrix when the longest first policy is used. Then (2.25) continues to hold for any fixed u , when Q is replaced by Q^{LF} . Therefore if the necessary condition (2.24) holds, ϵ can be taken as in Example 4a, and $Q^{LF}V(x) \leq -\epsilon(x_1 + x_2 + x_3) + \gamma$ for all x . So Corollary 2.2.4 implies that X is positive recurrent under the longer first policy, and $\overline{X}_1 + \overline{X}_2 + \overline{X}_3 \leq \frac{\gamma}{2\epsilon}$. (Note: We see that

$$Q^{LF}V(x) \leq \left(\sum_{i=1}^3 x_i \lambda_i \right) - m_1(x_1 \vee x_2) - m_2(x_2 \vee x_3) + \frac{\gamma}{2},$$

but for obtaining a bound on $\overline{X}_1 + \overline{X}_2 + \overline{X}_3$ it was simpler to compare to the case of random service allocation.)

2.3 Problems

2.1. Recurrence of mean zero random walks

(a) Suppose B_1, B_2, \dots is a sequence of independent, mean zero, integer valued random variables, which are bounded, i.e. $P[|B_i| \leq M] = 1$ for some M .

(a) Let $X_0 = 0$ and $X_n = B_1 + \dots + B_n$ for $n \geq 0$. Show that X is recurrent.

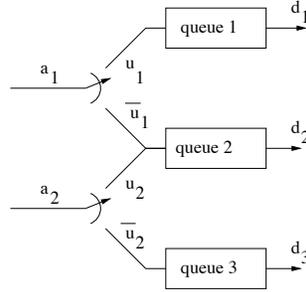
(b) Suppose $Y_0 = 0$ and $Y_{n+1} = Y_n + B_n + L_n$, where $L_n = -(Y_n + B_n)_+$. The process Y is a reflected version of X . Show that Y is recurrent.

2.2. Positive recurrence of reflected random walk with negative drift

Suppose B_1, B_2, \dots is a sequence of independent, integer valued random variables, each with mean $\overline{B} < 0$ and second moment $\overline{B^2} < +\infty$. Suppose $X_0 = 0$ and $X_{n+1} = X_n + B_n + L_n$, where $L_n = -(X_n + B_n)_+$. Show that X is positive recurrent, and give an upper bound on the mean under the equilibrium distribution, \overline{X} . (Note, it is not assumed that the B 's are bounded.)

2.3. Routing with two arrival streams

(a) Generalize Example 1.a to the scenario shown.

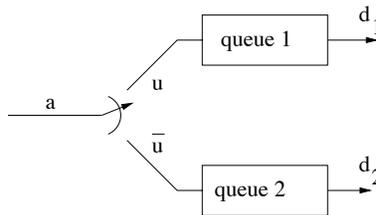


where $a_i, d_j \in (0, 1)$ for $1 \leq i \leq 2$ and $1 \leq j \leq 3$. In particular, determine conditions on a_1 and a_2 that insure there is a choice of $u = (u_1, u_2)$ which makes the system positive recurrent. Under those conditions, find an upper bound on $\overline{X}_1 + \overline{X}_2 + \overline{X}_3$, and select u to minimize the bound.

(b) Generalize Example 1.b to the scenario shown. In particular, can you find a version of route-to-shorter routing so that the bound found in part (a) still holds?

2.4. An inadequacy of a linear potential function

Consider the system of Example 1.b in the notes (a discrete time model, using the route to shorter policy, with ties broken in favor of queue 1, so $u = I_{\{x_1 \leq x_2\}}$):



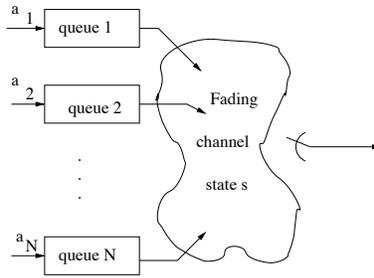
Assume $a = 0.7$ and $d_1 = d_2 = 0.4$. The system is positive recurrent. Explain why the function $V(x) = x_1 + x_2$ does *not* satisfy the Foster-Lyapunov stability criteria for positive recurrence, for any choice of the constant b and the finite set C .

2.5. Allocation of service

Prove the claim in Example 4a about the largest value of ϵ .

2.6. Opportunistic scheduling (Tassiulas and Ephremides [40])

Suppose N queues are in parallel, and suppose the arrivals to a queue i form an independent, identically distributed sequence, with the number of arrivals in a given slot having mean $a_i > 0$ and finite second moment K_i . Let $S(t)$ for each t be a subset of $E = \{1, \dots, N\}$ and $t \geq 0$. The random sets $S(t) : t \geq 0$ are assumed to be independent with common distribution w . The interpretation is that there is a single server, and in slot i , it can serve one packet from one of the queues in $S(t)$. For example, the queues might be in the base station of a wireless network with packets queued for N mobile users, and $S(t)$ denotes the set of mobile users that have working channels for time slot $[t, t + 1)$. See the illustration:



(a) Explain why the following condition is necessary for stability: For all $s \subset E$ with $s \neq \emptyset$,

$$\sum_{i \in s} a_i < \sum_{B: B \cap s \neq \emptyset} w(B) \quad (2.27)$$

(b) Consider u of the form $u = (u(i, s) : i \in E, s \subset E)$, with $u(i, s) \geq 0$, $u(i, s) = 0$ if $i \notin s$, and $\sum_{i \in E} u(i, s) = I_{\{s \neq \emptyset\}}$. Suppose that given $S(t) = s$, the queue that is given a potential service opportunity has probability distribution $(u(i, s) : i \in E)$. Then the probability of a potential service at queue i is given by $\mu_i(u) = \sum_s u(i, s)w(s)$ for $i \in E$. Show that under the condition (2.27), for some $\epsilon > 0$, u can be selected to that $a_i + \epsilon \leq \mu_i(u)$ for $i \in E$. (Hint: Apply the min-cut, max-flow theorem, given in Chapter 6 of the notes, to an appropriate graph.)

(c) Show that using the u found in part (b) that the process is positive recurrent.

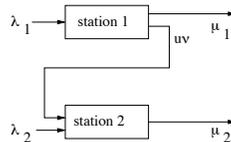
(d) Suggest a dynamic scheduling method which does not require knowledge of the arrival rates or the distribution w , which yields the same bound on the mean sum of queue lengths found in part (b).

2.7. Routing to two queues – continuous time model

Give a continuous time analog of Examples 1.a and 1.b. In particular, suppose that the arrival process is Poisson with rate λ and the potential departure processes are Poisson with rates μ_1 and μ_2 .

2.8. Stability of two queues with transfers

Let $(\lambda_1, \lambda_2, \nu, \mu_1, \mu_2)$ be a vector of strictly positive parameters, and consider a system of two service stations with transfers as pictured.



Station i has Poisson arrivals at rate λ_i and an exponential type server, with rate μ_i . In addition, customers are transferred from station 1 to station 2 at rate $u\nu$, where u is a constant with $u \in U = [0, 1]$. (Rather than applying dynamic programming here, we will apply the method of Foster-Lyapunov stability theory in continuous time.) The system is described by a continuous-time Markov process on \mathbb{Z}_+^2 with some transition rate matrix Q . (You don't need to write out Q .)

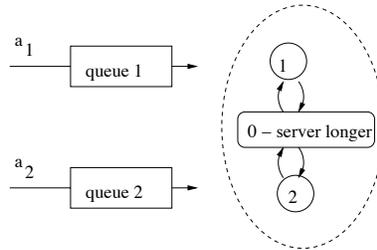
(a) Under what condition on $(\lambda_1, \lambda_2, \nu, \mu_1, \mu_2)$ is there a choice of the constant u such that the Markov process describing the system is positive recurrent?

(b) Let V be the quadratic Lyapunov function, $V(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2}$. Compute the drift function QV .

(c) Under the condition of part (a), and using the moment bound associated with the Foster-Lyapunov criteria, find an upper bound on the mean number in the system in equilibrium, $\overline{X}_1 + \overline{X}_2$. (The smaller the bound the better.)

2.9. Stability of a system with two queues and modulated server

Consider two queues, queue 1 and queue 2, such that in each time slot, queue i receives a new packet with probability a_i , where $0 < a_1 < 1$ and $0 < a_2 < 1$. Suppose the server is described by a three state Markov process, as shown.



If the server process is in state i for $i \in \{1, 2\}$ at the beginning of a slot, then a potential service is given to station i . If the server process is in state 0 at the beginning of a slot, then a potential service is given to the longer queue (with ties broken in favor of queue 1). Then during the slot, the server state jumps with the probabilities indicated. (Note that a packet can arrive and depart in one time slot.) For what values of a_1 and a_2 is the process stable? Briefly explain your answer (but rigorous proof is not required).

Chapter 3

Queues with General Interarrival Time and/or Service Time Distributions

3.1 The M/GI/1 queue

An M/GI/1 queue is one in which the arrival process is memoryless (in fact, Poisson, with some parameter $\lambda > 0$), the service times X_1, X_2, \dots of successive customers are independent with a general probability distribution function B , and there is a single server. The number of customers in the system at time t is denoted $N(t)$. A sample path of N is illustrated in Figure 3.1. For ease of notation, it is assumed that customers are served in the order in which they arrive. This is known as the First-Come, First-Served (FCFS) service discipline. As indicated in the figure, we define the following auxiliary variables:

C_n denotes the n th customer

q_n is the number in the system just after C_n departs, equilibrium distribution ($d_k : k \geq 0$)

q'_n is the number in the system just before C_n arrives, equilibrium distribution ($r_k : k \geq 0$)

v_n is the number of customers which arrive while C_n is being served

Our goal will be to compute the distribution of $N(t)$ in equilibrium.

Note that the random variable $N(t)$ for a fixed time t does not determine how long the customer in the server (if any) at time t has been in the server, whereas the past history ($N(s) : s \leq t$) does determine this time-in-service. The time-in-service of the customer in the server is relevant to the distribution of the future of the process N , unless the service time distribution is exponential. Thus, unless the service time distribution is exponential, ($N(t) : t \geq 0$) is not Markov.

Suppose for now that equilibrium distributions exist. Since the arrivals form a Poisson sequence, the equilibrium distribution of N is the same as the distribution $r = (r_k)$ seen by arrivals. Since as most one arrival and one departure can occur at a time, $r = d$ (see exercise). Thus, $r_k = p_k = d_k$ for all k .

Consider the evolution of ($q_n : n \geq 0$). Just after C_{n+1} begins service, there are $q_n + I_{\{q_n=0\}}$ customers in the system. The term $I_{\{q_n=0\}}$ accounts for the arrival of C_{n+1} in case the system is

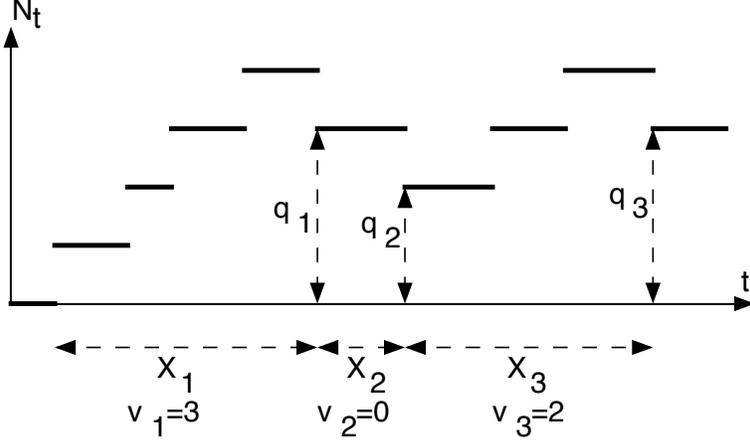


Figure 3.1: A typical sample path for an M/GI/1 queueing system

empty upon the departure of C_n . During the period C_{n+1} is served, there are v_{n+1} arrivals. This leads to the recursion

$$q_{n+1} = q_n - I_{\{q_n \neq 0\}} + v_{n+1} \quad (3.1)$$

The variables v_n are independent and identically distributed with common z -transform

$$V(z) = E[z^{v_n}] = \int_0^\infty \left[e^{-\lambda x} \sum_{k=0}^\infty \frac{(\lambda x z)^k}{k!} \right] dB(x) \quad (3.2)$$

$$= \int_0^\infty e^{-\lambda x + \lambda x z} dB(x) \quad (3.3)$$

$$= B^*(\lambda - \lambda z), \quad (3.4)$$

and mean $V'(1) = -\lambda B^*(0) = \lambda \bar{X} = \rho$. Thus, the sequence $(q_n : n \geq 0)$ is Markov with the one step transition probability matrix (letting $(\alpha_k : k \geq 0)$ denote the distribution of the v_n):

$$P = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ & \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \quad (3.5)$$

Due to the nearly upper triangular structure of the matrix P it is possible to recursively find the elements of p as a multiple of p_0 , and then find p_0 at the end by normalizing the sum to one. Another approach is to work with z -transforms, writing the equation $p = pP$ as $zP(z) = [P(z) + p_0(z-1)]V(z)$. Solve this to obtain

$$P(z) = \frac{p_0(z-1)V(z)}{z - V(z)} \quad (3.6)$$

To determine the constant p_0 use the fact that $\lim_{z \rightarrow 1} P(z) = 1$ (use l'Hospital's rule and the fact $V'(1) = \lambda \bar{X} = \rho$) to obtain $p_0 = 1 - \rho$. Therefore,

$$P(z) = \frac{(1 - \rho)(z-1)V(z)}{z - V(z)}, \quad (3.7)$$

which is the celebrated formula of Polleccek and Khinchine for the M/GI/1 system. Computation of the derivatives at one of $P(z)$, again using l'Hospital's rule, yields the moments $\overline{N^k}$ of the distribution of the number in system. In particular,

$$\overline{N} = \rho + \frac{\lambda^2 \overline{X^2}}{2(1-\rho)}. \quad (3.8)$$

To obtain the mean number in the queue, simply subtract the mean number in the server, ρ . To obtain the mean waiting time (in queue or system) apply Little's law to the corresponding mean number in system. For example, the mean waiting time in the *queue* is

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)}. \quad (3.9)$$

It is remarkable the W depends only on λ , \overline{X} and $\overline{X^2} = (\overline{X})^2 + Var(X)$. Note that W is increasing in the variance of X , for fixed mean \overline{X} and arrival rate λ .

A simple alternative derivation of (3.9) is based on renewal theory. Suppose the order of service is first-come, first-served. The mean waiting time W of a typical arrival is the same as the mean amount of work in the system at the time of arrival, which by the PASTA property is the same as the mean amount of work in the system in equilibrium. The mean work in the system can be decomposed to the mean work in the server plus the mean work in the queue. Thus,

$$W = R + N^Q \overline{X}, \quad (3.10)$$

where

$$R = \rho \left(\frac{\overline{X^2}}{2\overline{X}} \right), \quad (3.11)$$

which is the product of the probability the server is busy (ρ) times the mean residual service time of a customer in service, which is determined by renewal theory. The other term on the right of (3.10) accounts for the portion of delay caused by the customers that are in the queue at the time of an arrival. Applying Little's law, $N^Q \overline{X} = \lambda W \overline{X} = \rho W$, substituting into (3.10) and solving yields $W = R/(1-\rho)$, which is equivalent to the desired formula (3.9).

3.1.1 Busy Period Distribution

Call the customers that arrive during the service interval of a given customer an offspring of that customer. The offspring of a customer, together with the offspring of the offspring, and so forth, is the set of descendants of the customer. The length of the busy period initiated by the first customer is equal to the sum of the service times of the customer itself and of all its descendants. The number of offspring of the first customer is v_1 , which, given that $X_1 = x$, has the Poisson distribution of mean λx .

The length of the busy period L can be expressed as the sum

$$L = X_1 + L_1 + L_2 + \dots + L_{v_1} \quad (3.12)$$

where X_1 is the service time of the first customer in the busy period, v_1 is the number of offspring of that customer, and L_i is the sum of the service time of the i th offspring of C_1 and the service

time of that offspring's descendants. Notice that the joint distribution of L_1, L_2, \dots, L_{v_1} is as if independent samples with the same distribution as L are drawn.

Condition on X_1 to obtain:

$$G^*(s) = E[e^{-sL}] = \int_0^\infty e^{-sx} \left(\sum_{n=0}^\infty G^*(s)^n e^{-\lambda x} \frac{(\lambda x)^n}{n!} \right) dB(x) \quad (3.13)$$

$$= \int_0^\infty e^{-sx} e^{-\lambda x + \lambda G^*(s)x} B(dx) \quad (3.14)$$

$$= B^*(s + \lambda - \lambda G^*(s)) \quad (3.15)$$

Obtain $g_1 = \bar{X}/(1-\rho)$. Can also get this by a renewal argument. Obtain also $g_2 = \bar{X}^2/(1-\rho)^3$.

3.1.2 Priority M/GI/1 systems

Suppose there are n independent arrival streams, each modeled as a Poisson process, with the i th having arrival rate λ_i . Suppose the streams are merged into a single-server queueing system, and that the service times of customers from the i th stream are identically distributed with the same distribution as a generic random variable X_i .

An issue for the system may be the service order. For example, for a pure priority service order, the server attempts to serve the available customer with the highest priority. Some finer distinctions are possible. If, for example, service of a lower priority customer is interrupted by the arrival of a higher priority customer, the service discipline is said to be preemptive priority. Otherwise it is nonpreemptive priority.

Does the mean waiting time of a customer (averaged over all the streams according to arrival rates) depend on the service discipline? The answer is yes, which is why mass service facilities (such as supermarket checkout lines) often give priority to customers with short service requirements.

The stream obtained by merging the arrival streams is again $M/GI/1$ type. The arrival rate of the combined stream is $\lambda = \lambda_1 + \dots + \lambda_n$, and the service time of a customer in the combined stream is distributed as the random variable $X = X_J$. Here the index J is chosen at random, with $P[J = j] = \lambda_j/\lambda$, and J, X_1, \dots, X_n are mutually independent. Note that

$$\bar{X} = \sum_i \frac{\lambda_i \bar{X}_i}{\lambda} \quad \text{and} \quad \overline{X^2} = \sum_i \frac{\lambda_i \overline{X_i^2}}{\lambda} \quad (3.16)$$

As long as customers are served in FCFS order, (3.9) is valid. The PASTA property and the FCFS service order implies that W is equal to the mean work in the system, where work is measured in units of time. Therefore,

$$\text{Mean work in system} = \frac{\lambda \overline{X^2}}{2(1-\rho)} = \frac{R}{1-\rho}. \quad (3.17)$$

where R is given by (3.11), which here can be written as

$$R = \frac{\lambda \overline{X^2}}{2} = \frac{\sum_{i=1}^n \lambda_i \overline{X_i^2}}{2}.$$

Although (3.17) was just derived under the assumption that the service order is FCFS, the terms involved do not depend on the order of service, as long as the server remains busy whenever there

is work to be done (in which case the system is said to be “work conserving”). Thus, (3.17) is valid for any work-conserving service discipline.

Assuming the required equilibria exist, the mean work in the *queue* composed of type i customers is $\overline{N_i X_i} = \lambda_i W_i \overline{X_i} = \rho_i W_i$ for each i , and the mean amount of work in the server is R . The mean work in the system is thus $(\sum_{i=1}^n \rho_i W_i) + R$. Equating this with the last expression in (3.17) yields that

$$\left(\sum_i \rho_i W_i\right) + R = \frac{R}{1 - \rho}, \quad (3.18)$$

which can be simplified to yield the “conservation of work” equation:

$$\sum_i \rho_i W_i = \frac{\rho R}{1 - \rho}. \quad (3.19)$$

This equation is true for any work-conserving service order for which the required equilibrium means exist.

In order to derive the mean waiting time of customers of stream i a particular service order must be specified. As mentioned above, if the order is FCFS, then the mean waiting time is the same for all streams, and is given by (3.9). Suppose now that the nonpreemptive priority discipline is used, with stream i having priority over stream j if $i \leq j$. Let W_i continue to denote the mean waiting time in queue for customers of stream i . Upon arrival to the queue, a customer from stream 1 need wait only for the completion of the customer in service (if any) and for the service of other stream 1 customers, if any, already in the queue. Thus, $W_1 = R + (\lambda_1 W_1)/\mu_1$, where by Little’s law we write λW_1 for the mean number of stream one customers in the queue, and R is the expected residual service time of the customer in service, if any:

$$R = \frac{\lambda \overline{X^2}}{2} = \frac{\sum_{i=1}^n \lambda_i \overline{X_i^2}}{2} \quad (3.20)$$

Solving for W_1 yields

$$W_1 = \frac{R}{1 - \rho_1}. \quad (3.21)$$

Next, argue that W_2 is equal to $R + (\lambda_1 W_1)/\mu_1 + (\lambda_2 W_2)/\mu_2 + \lambda_1 \overline{X_1} W_2$. The first three terms in this sum account for the residual service time and the service times of stream 1 and stream 2 customers already in the queue at the time of arrival of a stream 2 packet. The last term accounts for the expected wait due to type 1 customers that arrive while the typical arrival from stream 2 is in the queue. Solve for W_2 to find

$$W_2 = \frac{R + \rho_1 W_1}{1 - \rho_1 - \rho_2} = \frac{R}{(1 - \rho_1)(1 - \rho_1 - \rho_2)} \quad (3.22)$$

Finally, continuing by induction on i , obtain

$$W_i = \frac{R}{(1 - \rho_1 - \dots - \rho_{i-1})(1 - \rho_1 - \dots - \rho_i)} \quad (3.23)$$

for nonpreemptive priority order of service.

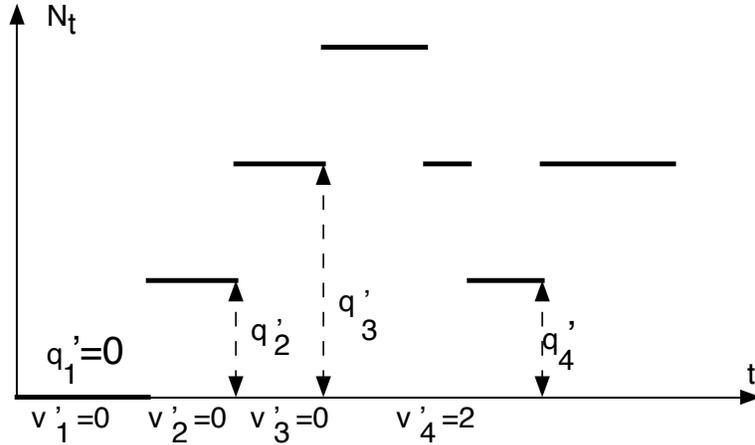


Figure 3.2: A GI/M/1 queue sample path

3.2 The GI/M/1 queue

A GI/M/1 queue is one in which the arrival process is a renewal process, service times are independent and exponentially distributed with a common mean, and there is a single server. Let A denote the cumulative distribution function and $1/\lambda$ the mean of the interarrival times, and let μ denote the parameter of the service times. The number of customers in the system at time t is denoted $N(t)$. A sample path of N is illustrated in Figure 3.2. It is assumed that customers are served in FCFS order.

It is useful to think of the service process as being governed by a Poisson process of rate μ . Jumps in the process denote potential service completions. A potential service completion becomes an actual service completion if there is at least one customer in the system at the time of the potential service completion. By the properties of the Poisson process and the exponential distribution, this view of the system is consistent with its distribution. Define the following auxiliary variables, which are indicated in Figure 3.2:

C_n denotes the n th customer

q'_n is the number in the system just before C_n arrives, equilibrium distribution ($r_k : k \geq 0$)

\hat{v}_n is the number of potential service completions which occur during the interarrival period just before the arrival of C_n .

Note that the random variable $N(t)$ for a fixed time t does not determine how long before time t the last customer arrival occurred, whereas the past history ($N(s) : s \leq t$) does determine the time-since-most-recent-arrival. The time-since-most-recent-arrival is relevant to the distribution of the future of the process N , unless the interarrival distribution is exponential. Thus, unless the arrival process is Poisson, ($N(t) : t \geq 0$) is not Markov. Nevertheless, the mean waiting time, and in fact the waiting time distribution, can be determined by focusing on the process ($q'_n : n \geq 1$).

Consider the evolution of ($q'_n : n \geq 0$). Just after C_n arrives, there are $q'_n + 1$ customers in the system. There are no more arrivals and potentially as many as q'_n departures in the period strictly

between arrival times of C_n and C_{n+1} . Therefore,

$$q'_{n+1} = (q'_n + 1 - \hat{v}_{n+1})_+ \quad (3.24)$$

The variables \hat{v}_n are independent and identically distributed with common z -transform $A^*(\mu - \mu z)$. and mean $-\mu A^{*'}(0) = \mu/\lambda = 1/\rho$. Thus, the sequence $(q'_n : n \geq 0)$ is Markov with the one step transition probability matrix (letting $(\beta_k : k \geq 0)$ denote the distribution of the \hat{v}_n):

$$P = \begin{bmatrix} 1 - \beta_0 & \beta_0 & 0 & 0 & \dots \\ 1 - \beta_0 - \beta_1 & \beta_1 & \beta_0 & 0 & \dots \\ 1 - \beta_0 - \beta_1 - \beta_2 & \beta_2 & \beta_1 & \beta_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \quad (3.25)$$

The equilibrium equation $r = rP$ can be written

$$r_k = \beta_0 r_{k-1} + \beta_1 r_k + \dots \quad k \geq 1 \quad (3.26)$$

Seek a solution of the form $r_k = (1 - \sigma)\sigma^k$ for some $\sigma > 0$. This yields the equation for σ :

$$\sigma = A^*(\mu - \mu\sigma) \quad (3.27)$$

The function $A^*(\mu - \mu\sigma)$ is strictly convex, value one and derivative $1/\rho$ at $\sigma = 1$. Thus, a solution σ with $0 < \sigma < 1$ exists if and only if $\rho < 1$.

If an equilibrium probability distribution $r = (r_k : k \geq 0)$ of (q'_k) exists, then the process is positive recurrent. Recall from the proof of Proposition 1.7.2, that for a given state, say k , the mean number of visits to state $k + 1$ per excursion from state k is r_{k+1}/r_k . Since the process can increase by only one at a time, in order that the process reach $k + 1$ before k given that it starts in k , it must take its first jump towards state $k + 1$. As soon as the process crosses below state k , it has no more opportunities to hit $k + 1$ before reaching k . By the shift invariance of the transition probabilities, it follows that the ratio r_{k+1}/r_k is the same for all k .

$$\frac{r_{k+1}}{r_k} = E[\text{number of visits to } k + 1 \text{ before return to } k | q'_0 = k] \quad (3.28)$$

This ratio does not depend on k . Thus, if any equilibrium distribution exists, it must be a geometric distribution.

Given $q'_n = k$, the waiting time in queue of C_n is the sum of k independently distributed Poisson random variables with parameter μ , which has Laplace transform $\mu/(\mu + s)$. Thus, the Laplace transform of the waiting time in equilibrium is

$$E[e^{-sW}] = (1 - \sigma) \sum_{k=0}^{\infty} \sigma^k \left(\frac{\mu}{s + \mu} \right)^k \quad (3.29)$$

$$= 1 - \sigma + \sigma \left[\frac{\mu(1 - \sigma)}{s + \mu(1 - \sigma)} \right] \quad (3.30)$$

Thus, the equilibrium waiting distribution is a mixture of the random variable that is identically zero, and an exponentially distributed random variable with parameter $\mu(1 - \sigma)$. Thus, $P[W > x] = \sigma \exp(-x\mu(1 - \sigma))$. and W has mean $\bar{W} = \sigma/((1 - \sigma)\mu)$. By Little's law, the mean number in the system is given by $\bar{N} = \lambda(\bar{W} + 1/\mu) = \rho/(1 - \sigma)$. The total time in the system, T , can be represented as $T = W + X$, where W is the waiting time, X is the service time, and W and X are mutually independent. The Laplace transform of the system time is given by $E[e^{-sT}] = E[e^{-sW}]E[e^{-sX}] = \frac{(1 - \sigma)\mu}{s + \mu(1 - \sigma)}$. That is, the total time in the system T is exponentially distributed with parameter $\mu(1 - \sigma)$.

3.3 The GI/GI/1 System

A GI/GI/1 queue is one in which the arrival process is renewal type, the service times of successive customers are independent and identically distributed, and there is a single server. It is assumed that customers are served in FCFS order.

C_n denotes the n th customer

x_n is the service time required for the n th customer

t_n is the time between the arrival of C_{n-1} and C_n

w_n is the waiting time in queue of the n th customer

$U(t)$ is the work in system, also known as the virtual waiting time, at time t

Take as the initial condition for the system that it is empty at time 0 and C_1 arrives at time t_1 . With these definitions, notice that $w_n = U(t_1 + \dots + t_n -)$.

A useful stochastic equation is derived next. Just after the arrival of customer C_n , the amount of work in the system is $w_n + x_n$, and the work begins to decrease at unit rate. Customer C_{n+1} arrives to find the system nonempty if and only if $w_n + x_n - t_{n+1} > 0$, and if this condition holds then $w_{n+1} = w_n + x_n - t_{n+1}$. Therefore, in general,

$$w_{n+1} = (w_n + u_n)_+ \quad (3.31)$$

where $u_n = x_n - t_{n+1}$. The random variables of the sequence $(u_n : n \geq 1)$ are independent and identically distributed, because the same is true of the two independent sequences $(x_n : n \geq 1)$ and $(t_n : n \geq 1)$. Then $w_1 = 0$, and by repeated use of (3.31),

$$\begin{aligned} w_n &= \max\{0, w_{n-1} + u_{n-1}\} \\ &= \max\{0, u_{n-1} + \max\{0, w_{n-2} + u_{n-2}\}\} \\ &= \max\{0, u_{n-1}, u_{n-1} + u_{n-2} + w_{n-2}\} \\ &= \max\{0, u_{n-1}, u_{n-1} + u_{n-2}, u_{n-1} + u_{n-1} + u_{n-2} + w_{n-3}\} \\ &\quad \vdots \\ &= \max\{0, u_{n-1}, u_{n-1} + u_{n-2}, \dots, u_{n-1} + \dots + u_1\} \end{aligned}$$

Let us next consider the question of stability. Define a random variable \tilde{w}_n by

$$\tilde{w}_n = \max\{0, u_1, u_1 + u_2, \dots, u_1 + \dots + u_{n-1}\}, \quad (3.32)$$

which is the same as w_n , but with the sequence (u_1, \dots, u_{n-1}) reversed in time. Since time reversal preserves the distribution of the u_n sequence, it is clear that for each n , w_n has the same distribution as \tilde{w}_n . The sequence \tilde{w}_n is nondecreasing in n , so that the limit

$$\begin{aligned} \tilde{w}_\infty &= \lim_{n \rightarrow \infty} \tilde{w}_n \\ &= \max\{0, u_1, u_1 + u_2, \dots\} \end{aligned}$$

exists (with values in $(0, \infty]$) and is equal to the maximum of the partial sums of the u_n sequence as shown. Since convergence with probability one implies convergence in distribution, it follows that the variables (\tilde{w}_n) converge in distribution to \tilde{w}_∞ . Since convergence in distribution is determined by marginal distributions only, it follows that

$$\lim\text{-in-dist}_{n \rightarrow \infty} w_n = \tilde{w}_\infty \quad (3.33)$$

Suppose that the x_n and t_n have finite means, so that u_n also has a finite mean and the strong law of large numbers applies: $\lim_{n \rightarrow \infty} (u_1 + \dots + u_n)/n = E[u_1]$ with probability one. If $E[u_n] < 0$, it follows that $\lim_{n \rightarrow \infty} (u_1 + \dots + u_n) = -\infty$ with probability one, which implies that $P[\tilde{w}_\infty < \infty] = 1$. The arrival rate is $\lambda = 1/E[t_1]$ and the service rate is $\mu = 1/E[x_1]$. Letting $\rho = \lambda/\mu$ as usual, it has just been proved that if $\rho < 1$, then the variables w_n have a limiting distribution with no probability mass at $+\infty$.

There are techniques for computing the distribution of waiting time. One is based on Wiener-Hopf spectral factorization, and another on certain identities due to F. Sipitser. However, some simple bounds on the waiting time distribution, presented in the next section, are often more useful in applications.

3.4 Kingman's Bounds for GI/GI/1 Queues

The elegant bounds, due to J.F.C. Kingman, on the mean and tail probabilities for the waiting time in a GI/GI/1 system are given in this section. Let σ_z^2 denote the variance of a random variable z .

Proposition 3.4.1 (*Kingman's moment bound*) *The mean waiting time W for a GI/GI/1 FCFS queue satisfies*

$$W \leq \frac{\lambda(\sigma_x^2 + \sigma_t^2)}{2(1 - \rho)} \quad (3.34)$$

Proof. Let w denote a random variable with the distribution of w_∞ and let u denote a random variable independent of w with the same distribution as the u_n 's. Taking $n \rightarrow \infty$ in (3.31) yields that $w \stackrel{d}{=} (w + u)_+$, where " $\stackrel{d}{=}$ " denotes equality in distribution. In particular, $E[w] = E[(w + u)_+]$ and $E[w^2] = E[(w + u)_+^2]$. Clearly

$$w + u = (w + u)_+ - (w + u)_-, \quad (3.35)$$

where $(w + u)_- = -\min\{0, w + u\}$. Taking expectations of each side of (3.35) yields $E[u] = -E[(w + u)_-]$. Squaring each side of (3.35), using the fact $(w + u)_+(w + u)_- = 0$ and taking expectations yields

$$E[w^2] + 2E[w]E[u] + E[u^2] = E[(w + u)_+^2] + E[(w + u)_-^2],$$

or equivalently

$$\begin{aligned} E[w] &= \frac{E[u^2] - E[(w + u)_-^2]}{-2E[u]} \\ &= \frac{\sigma_u^2 - \sigma_{(w+u)_-}^2}{-2E[u]}. \end{aligned}$$

Noting that $\sigma_{(w+u)_-}^2 \geq 0$ and $\sigma_u^2 = \sigma_x^2 + \sigma_t^2$ yields the desired inequality. \blacksquare

Exercise: Compare Kingman's moment bound to the exact expression for W in case the interarrival times are exponentially distributed. Solution: Then $\sigma_t^2 = \frac{1}{\lambda^2}$ so the bound becomes

$$E[w] \leq \frac{\lambda(\sigma_x^2 + \frac{1}{\lambda^2})}{2(1 - \rho)}$$

whereas by the Polleczeck and Khinchine mean formula for an M/GI/1 queuing system

$$E[w] = \frac{\lambda(\sigma_x^2 + \frac{1}{\mu^2})}{2(1 - \rho)}.$$

Next Kingman's tail bound is given. Let w_n denote the waiting time in queue of the n^{th} customer in a GI/GI/1 system. Then w_{n+1} has the same probability distribution as $\max_{0 \leq k \leq n} u_1 + \dots + u_k$, where u_k is the service time required by the k^{th} customer minus the length of time between the k^{th} and $k+1^{\text{th}}$ arrivals. By assumption, the random variables u_1, u_2, \dots are identically distributed. Suppose $E[u_n] < 0$ and that $\phi(\theta) = E[\exp(\theta u_n)] < +\infty$ for some $\theta > 0$. Let $\theta^* = \sup\{\theta : \phi(\theta) \leq 1\}$.

Proposition 3.4.2 (Kingman's tail bound) For $n \geq 0$ and $b \geq 0$, $P[w_{n+1} \geq b] \leq \exp(-\theta^* b)$.

Proof. Choose any $\theta > 0$ such that $\phi(\theta) \leq 1$. Let $M_0 = 1$ and $M_k = \exp(\theta(u_1 + \dots + u_k))$. Then the probability in question can be expressed as $P[w_{n+1} \geq b] = P[(\max_{0 \leq k \leq n} M_k) \geq \exp(\theta b)]$. It is easy to check that $E[M_{k+1}|M_k, \dots, M_0] \leq M_k$, so by Lemma 3.4.3 below,

$$P[\max_{1 \leq k \leq n} M_k \geq \exp(\theta b)] \leq E[M_0]/\exp(\theta b) = \exp(-\theta b). \quad (3.36)$$

Thus, $P[w_{n+1} \geq b] \leq \exp(-\theta b)$ for $0 < \theta < \theta^*$, which implies the proposition. \blacksquare

Lemma 3.4.3 Let M_0, M_1, \dots be nonnegative random variables such that $E[M_{k+1}|M_k, \dots, M_0] \leq M_k$ for $k \geq 0$ (i.e. M is a nonnegative supermartingale). Then for any $n \geq 0$, $P[(\max_{0 \leq k \leq n} M_k) \geq \gamma] \leq E[M_0]/\gamma$.

Proof. Let $\tau = \min\{k \geq 0 : M_k \geq \gamma\}$, with the convention that $\tau = +\infty$ if the indicated set is empty. Then $(\max_{0 \leq k \leq n} M_k) \geq \gamma$ if and only if $M_{\tau \wedge n} \geq \gamma$, where $a \wedge b$ denotes the minimum of a and b . This fact and the simple "Markov inequality" implies that $P[(\max_{0 \leq k \leq n} M_k) \geq \gamma] = P[M_{\tau \wedge n} \geq \gamma] \leq E[M_{\tau \wedge n}]/\gamma$. Since $E[M_{\tau \wedge n}] = E[M_0]$ for $n = 0$, the proof can be completed by showing that $E[M_{\tau \wedge n}]$ is nonincreasing in n . To that end, note that $M_{\tau \wedge (n+1)} - M_{\tau \wedge n} = (M_{n+1} - M_n)I_{\{\tau > n\}}$, so

$$\begin{aligned} E[M_{\tau \wedge (n+1)}] - E[M_{\tau \wedge n}] &= E[(M_{n+1} - M_n)I_{\{\tau > n\}}] \\ &= E[E[(M_{n+1} - M_n)I_{\{\tau > n\}}|M_n, \dots, M_0]] \\ &= E[E[(M_{n+1} - M_n)|M_n, \dots, M_0]I_{\{\tau > n\}}] \\ &\leq 0, \end{aligned}$$

where we use the fact that the event $\{\tau > n\}$ is determined by M_0, \dots, M_n . The lemma is proved. \blacksquare

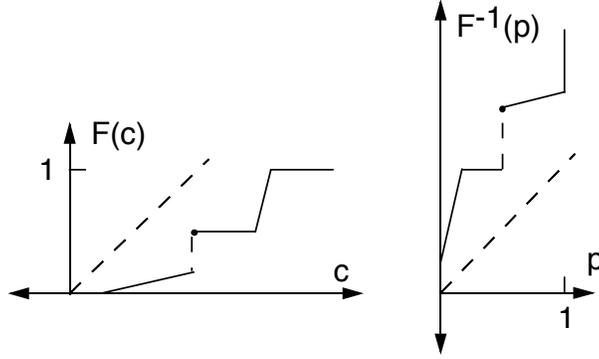


Figure 3.3: A distribution function and its inverse

Exercise: Compare Kingman's tail bound to the exact tail probabilities in case the service time distribution is exponential. Solution: In that case $\phi(\theta) = \frac{\mu}{\mu - \theta} A^*(\theta)$ so the equation for θ^* becomes $A^*(\theta) = 1 - \frac{\theta}{\mu}$. With the change of variable $\theta = \mu(1 - \sigma)$ this equation becomes the same as the equation for σ in the analysis of a GI/M/1 queueing system. Thus, θ^* in Kingman's bound is equal to $\mu(1 - \sigma)$ where σ is the value determined in the analysis of a GI/M/1 system. Thus, $P[w_\infty \geq b] = \sigma \exp(-\theta^* b)$, showing that Kingman's bound gives the same exponential behavior as the actual tail probabilities.

3.5 Stochastic Comparison with Application to GI/GI/1 Queues

Before describing the most basic part of the theory of stochastic comparison theory, we shall review the problem of generating a random variable with a given cumulative distribution function. Suppose F is a probability distribution function and that U is a random variable uniformly distributed on the interval $[0, 1]$. The inverse function of F is defined by

$$F^{-1}(p) = \sup\{x : F(x) \leq p\} \quad (3.37)$$

and an example is pictured in Figure 3.5. The graph of F^{-1} is the graph of F reflected about the line of slope one through the origin. Let $X = F^{-1}(U)$. Then for any constant c , $P[X \leq c] = P[F^{-1}(U) \leq c] = P[U \leq F(c)] = F(c)$, so that X has distribution F_c .

Exercise 1. a) Find and sketch F^{-1} assuming that F is the distribution function of an exponential random variable with parameter α . b) Repeat part a) in case F corresponds to a random variable uniformly distributed over the set $\{1, 2, 3, 4, 5, 6\}$. Simultaneously sketch F and F^{-1} as is done in Figure 3.5. Is F^{-1} right-continuous? c) Show that if $F(c) \leq G(c)$ for all c , then $F^{-1}(p) \geq G^{-1}(p)$ for all p .

Definition 3.5.1 Let F and G be two distribution functions. We say F stochastically dominates G , and write $F \succ G$, if $F(c) \leq G(c)$ for all constants c . If X and Y are two random variables, we say that X stochastically dominates Y , and write $X \succ Y$, if F_X stochastically dominates F_Y , where F_X is the distribution function of X and F_Y is the distribution function of Y .

Note that whether $X \succ Y$, depends only on the individual distributions of X and Y , and otherwise *does not depend on the joint distribution of X and Y* . In fact, X and Y don't even need to be random variables on the same probability space.

The fundamental proposition of stochastic comparison theory is the following:

Proposition 3.5.2 *$X \succ Y$ if and only if there is a pair of random variables (\tilde{X}, \tilde{Y}) defined on some probability space such that the following three conditions hold:*

$$\tilde{X} \stackrel{d}{=} X \quad (\text{the notation “}\stackrel{d}{=}\text{” means the variables have the sam probability distribution.})$$

$$\tilde{Y} \stackrel{d}{=} Y$$

$$P[\tilde{X} \geq \tilde{Y}] = 1.$$

Proof. If $X \succ Y$, let U be a uniform random variable and let $\tilde{X} = F_X^{-1}(U)$ and $\tilde{Y} = F_Y^{-1}(U)$. Then (\tilde{X}, \tilde{Y}) satisfies the required three conditions.

Conversely, if there exists (\tilde{X}, \tilde{Y}) satisfying the three conditions then for any constant c , the event $\{\tilde{X} \leq c\}$ is a subset of the event $\{\tilde{Y} \leq c\}$ (possibly ignoring an event of probability zero). Therefore, $F_X(c) = F_{\tilde{X}}(c) \leq F_{\tilde{Y}}(c) = F_Y(c)$, so that indeed $X \succ Y$. ■

A useful method for deriving bounds on probability distributions, called the *stochastic comparison* method, or *coupling* method, consists of applying Proposition 3.5.2. Note that the random variables X and Y may have little to do with each other. They may be independent, or even defined on different probability spaces. Variables \tilde{X} and \tilde{Y} defined on a common statespace, with $\tilde{X} \stackrel{d}{=} X$ and $\tilde{Y} \stackrel{d}{=} Y$ are called coupled versions of X and Y . Thus, to establish that $X \succ Y$, it suffices to show the existence of coupled versions \tilde{X} and \tilde{Y} such that $P[\tilde{X} \geq \tilde{Y}] = 1$.

Example Let w_n , for $n \geq 1$, denote the sequence of waiting times for the customers in a GI/GI/1 queue, which is empty at the time of arrival of customer one. Then the random variables $\tilde{w}_1, \tilde{w}_2, \dots$ defined by (3.32) are coupled versions of w_1, w_2, \dots . Since the random variables \tilde{w}_n are nondecreasing in n with probability one, conclude that $w_1 \prec w_2 \prec \dots$. That is, the waiting time of customer C_n stochastically increases with n .

Example As a related example, let us demonstrate that the waiting time in a GI/GI/1 queue is an increasing function of the service time distribution in the sense of stochastic comparison. That is, suppose there are two queueing systems, called system 1 and system 2. Suppose that both systems are initially empty and that the sequence of arrival times of the two systems have identical distributions. Suppose that both systems use the *FIFO* service order and that the service times of the customers in system i , ($i = 1$ or $i = 2$) are independent and identically distributed with CDF $B^{(i)}$. Let $W_n^{(i)}$ denote the waiting time of the n^{th} customer in system i .

Suppose that $B^{(1)} \succ B^{(2)}$. We will establish that for each n , $W_n^{(1)} \succ W_n^{(2)}$. Consider a probability space on which there is defined a sequence of variables (T_1, T_2, \dots) which has the same distribution as the sequence of interarrival times for either of the two queueing systems being compared. On the same probability space, for each n , let $(X_n^{(1)}, X_n^{(2)})$ be a pair of random variables such that $X_n^{(1)}$ has CDF $B^{(1)}$, $X_n^{(2)}$ has CDF $B^{(2)}$, and $P[X_n^{(1)} \geq X_n^{(2)}] = 1$. Such a pair can be constructed according to Proposition 3.5.2. Moreover, the pairs in the sequence $(X_n^{(1)}, X_n^{(2)})_{n \geq 1}$ can be constructed

to be mutually independent, and also to be independent of (T_1, T_2, \dots) . For $i = 1$ or $i = 2$, define $\tilde{W}_n^{(i)}$ to be the waiting time of the n^{th} customer in the queueing system in which (T_1, T_2, \dots) is the interarrival time sequence and $(B_1^{(i)}, B_2^{(i)}, \dots)$ is the sequence of customer service times. Clearly $\tilde{W}_n^{(i)}$ and $W_n^{(i)}$ have the same distribution, so it suffices to prove that $P[\tilde{W}_n^{(1)} \geq \tilde{W}_n^{(2)}] = 1$. However, the fact $\tilde{W}_1^{(1)} = \tilde{W}_1^{(2)} = 0$, the update equations $\tilde{W}_{n+1}^{(i)} = (\tilde{W}_n^{(i)} - T_{n+1} + X_n^{(i)})_+$, and the fact $P[X_n^{(1)} \geq X_n^{(2)}] = 1$ imply that $P[\tilde{W}_n^{(1)} \geq \tilde{W}_n^{(2)}] = 1$ for all $n \geq 1$ by induction on n . Applying Proposition 3.5.2 again, we conclude that $W_n^{(1)} \succ W_n^{(2)}$ for all n , as advertised.

Exercise 2 a) Show that $X \succ Y$ if and only if for every nondecreasing function ϕ , $E[\phi(X)] \geq E[\phi(Y)]$. (Another type of stochastic ordering requires the inequality to hold only for convex, nondecreasing functions ϕ .) *b)* Let X be uniformly distributed on $[0, 1]$, let Y be exponentially distributed with $E[Y] = 1$, and let Z have the Poisson distribution with $E[Z] = 1$. Identify any stochastic orderings that appear between pairs of these random variables (e.g., is $X \succ Y$?). *c)* Suppose $X \succ Y$ and $E[X] = E[Y]$. Prove that X and Y have the same distribution.

3.6 GI/GI/1 Systems with Server Vacations, and Application to TDM and FDM

In a GI/GI/1 system with server vacations, whenever the server finishes serving a customer or returns from a vacation, and finds the system empty, the server goes on another vacation. A vacation is a period of time in which the server does not serve customers, even if one or more customers arrive during the period. An effective method for accounting for the effect of vacations is to compare the sample paths of a system operating without vacations to one with vacations. The following variables will be used.

- w_n^o is the waiting time in queue of the n^{th} customer in the system *without* vacations.
- w_n is the waiting time in queue of the n^{th} customer in the system *with* vacations.
- t_1 is the time of the first arrival and, for $n \geq 1$, t_{n+1} is the time between the n^{th} and $n + 1^{\text{th}}$ arrivals. These variables are used in both systems. The variables t_1, t_2, \dots are assumed to be independent and identically distributed.
- x_n is the service time required by the n^{th} customer. These variables are used in both systems. The variables x_1, x_2, \dots are assumed to be independent and identically distributed.
- V_n is the length of the n^{th} vacation period for the system with vacations. The variables V_n for $n \geq 1$ are assumed to be independent, and identically distributed.

The two systems both start empty, and in the system with vacations, the server initiates a vacation at time zero with duration V_0 . Roughly speaking, the main difference between the two systems is that when the n^{th} customer begins service in the system with vacations, the total idle time of the server is of the form $V_0 + \dots + V_k$ for some k . In fact, k is the minimum integer, subject to the constraint that the idle time in the system with vacations is greater than or equal to the idle time in the system without vacations. This idea is expressed concisely in the lemma below.

Let $\gamma(u)$ denote the residual lifetime process for the renewal process determined by V_0, V_1, \dots . That is, $\gamma(u) = V_0 + \dots + V_{l(u)} - u$, where $l(u) = \min\{k : V_0 + \dots + V_k \geq u\}$. Also, let I_n^o

denote the total idle time, in the system without vacations, up to the time the n^{th} customer begins (or ends) service.

Lemma 3.6.1 For $n \geq 1, w_n = w_n^o + \gamma(I_n^o)$.

Proof. Argue by induction on n . Since $w_1 = \gamma(I_1^o)$ and $w_1^o = 0$, the lemma is true for $n = 1$. Suppose the lemma is true for n , and argue that it is true for $n + 1$ as follows. Let x denote the amount of time (possibly zero) the server in the system without vacations is idle between services of the n^{th} and $n + 1^{\text{th}}$ customers. Therefore, $I_{n+1}^o = I_n^o + x$, and two cases are considered: (1) If $x \leq w_n - w_n^o$ then the server in the system with vacations will not take a vacation between service of the n^{th} and $n + 1^{\text{th}}$ customers. Therefore $w_{n+1} - w_{n+1}^o = w_n - w_n^o - x = \gamma(I_n^o) - x = \gamma(I_n^o + x) = \gamma(I_{n+1}^o)$, as was to be proved. (2) If $x > w_n - w_n^o$ then at the time of arrival of the $n + 1^{\text{th}}$ customer, both systems are empty and both have been idle for I_{n+1}^o time units. Therefore, $w_{n+1} = \gamma(I_{n+1}^o)$ and $w_{n+1}^o = 0$, so again $w_{n+1} - w_{n+1}^o = \gamma(I_{n+1}^o)$. The induction step is complete, and the lemma is proved. ■

By renewal theory, if the distribution of V_1 is nonlattice type, the random variable $\gamma(u)$ converges in distribution as $u \rightarrow \infty$, where the limit has probability density function (p.d.f.) f_γ given by

$$f_\gamma(c) = \frac{P[V_1 \geq c]}{E[V_1]}. \quad (3.38)$$

It is useful to skip the question of convergence to equilibrium by choosing the distribution of V_0 to make $(\gamma(u) : u \geq 0)$ a stationary random process (rather than only asymptotically stationary). Thus, assume that V_0 has p.d.f. f_γ . Consequently, $\gamma(u)$ has distribution function f_γ for all $u \geq 0$. Since $(\gamma(u) : u \geq 0)$ is independent of (w_n^o, I_n^o) , the conditional density of $\gamma(I_n^o)$ given (w_n^o, I_n^o) is again f_γ . Thus (w_n^o, I_n^o) and $\gamma(I_n^o)$ are mutually independent. The following proposition is proved.

Proposition 3.6.2 Suppose V_0 has p.d.f. f_γ . Then w_n^o and $\gamma(I_n^o)$ are independent, $\gamma(I_n^o)$ has p.d.f. f_γ , and $w_n = w_n^o + \gamma(I_n^o)$.

Since the k^{th} moment of a random variable with p.d.f. f_γ is $m_{k+1}/((k+1)m_1)$ where $m_j = E[V_1^j]$, the following corollary is immediate from the Proposition.

Corollary 3.6.3 Suppose V_0 has p.d.f. f_γ . Then $E[w_n] = E[w_n^o] + m_2/(2m_1)$ and $\text{Var}(w_n) = \text{Var}(w_n^o) + m_3/(3m_1) - (m_2/(2m_1))^2$.

If V_0 has some arbitrary distribution, the Proposition still holds in a limiting sense, under an additional assumption to prevent periodic behavior. For example, if $E[x_1] < E[t_1] < \infty$ and if V_1 or $x_1 - t_1$ is nonlattice type, then the pair $(\gamma(I_n^o), w_n^o)$ converges in distribution as $n \rightarrow \infty$, and the limit $(\tilde{\gamma}, \tilde{w})$ has independent coordinates, with the first having p.d.f. f_γ .

As an example, one can model a time-slotted system in which the time axis is divided into frames of length M , for some constant M , and service can only begin at the beginning of a frame. To that end, choose the service time random variable x_1 to be a possibly-random integer multiple of M , and suppose that the vacations are all M units in duration. Then f_γ is the uniform density on the interval $[0, M]$. By the corollary, the mean waiting time in the system is $M/2$ time units larger than the mean waiting time in the system with the same service time distribution, without the constraint that service begin at the beginning of frames.

Continuing this example further, if M is an integer, we can think of a frame as being composed of M unit length slots. The waiting time in the slotted system is the same as that for a stream traversing a Time Division Multiplexed (TDM) communication link, in which the stream is served during one slot per frame. In the TDM system there are M arrival streams, all multiplexed onto a single line, and the queuing processes for the M streams are not related. When the vacations are dropped, the model represents a Frequency Division Multiplexed (FDM) communication link of the same total capacity. In an FDM multiplexed link, the M streams again do not interact. The mean waiting times are thus related by $W_{TDM} = W_{FDM} + M/2$. However, for the TDM system, the actual service time of a packet is $M - 1$ time units less than the service time in an FDM system, so the total mean system time for packets in the two systems are related by $T_{TDM} = T_{FDM} - M/2 + 1$. Note that there is not much difference between the delay in TDM and FDM systems, especially for heavily loaded systems.

Finally, to specialize the example further, if the traffic arrival process is Poisson with rate λ/M packets/slot and if each packet is served in M slots in the FDM system and in one slot in the TDM system (with only one out of every M slots being available), then the FDM system is an M/GI/1 system with constant service times (i.e. an M/D/1 system). The P-K mean formula thus yields $W_{FDM} = \lambda M / (2(1 - \lambda))$, and so $W_{TDM} = W_{FDM} + M/2 = M / (2(1 - \lambda))$.

3.7 Effective Bandwidth of a Data Stream

There has been an extensive effort since the inception of packet-switched communication networks to characterize the traffic carried by networks. The work aims to account for the bursty nature of many data sources. This section describes one approach for defining the effective bandwidth of a data stream. The approach is based on the theory of large deviations.

Recently, there has been a keen interest in accounting for the observations of many studies of traffic in real networks, which indicate that traffic streams exhibit self-similar behavior. That is, the random fluctuations in the arrival rate of packets appears to be nearly statistically the same on different time scales, ranging over several orders of magnitude. We touch on this development briefly in the context of the effective bandwidth of a self-similar Gaussian source. An extensive annotated bibliography on the subject is given in [43].

One of the primary goals of information theory is to identify the “effective information rate” of a data source. The entropy or the rate-distortion function of a data source may be thought of as such. The theory of effective bandwidth, described in this section, has a similar goal. Another connection between the theory of effective bandwidth of data streams and information theory is that much of the theory of effective bandwidth is based on large deviations theory, which is a close cousin to Shannon’s theory of information. Moreover, there is possibly some yet to be discovered direct connection between the theory of effective bandwidth and Shannon’s theory of information. For example, perhaps an “effective-bandwidth vs. distortion” function can be computed for some nontrivial sources.

A major way that the theory of effective bandwidth differs from the Shannon theory is that it treats the flow of data bits as it would the flow of anything else, such as a fluid. The values of the bits are not especially relevant. The idea is that individual connections or data streams carried by a network may be variable in nature. The data rate may be variable in a statistical sense, in that the rate produced by the stream or connection is not a priori known and can possibly be considered to be random. It may also be varying in time: such is the case for data streams produced by variable

rate source coders. Suppose many variable data streams are multiplexed together onto a line with a fixed capacity, where capacity is measured in bits per second. Because of statistical multiplexing, the multiplexer has less work to do than if all the data streams were sending data at the peak rate all the time. Therefore a given data stream has an effective bandwidth (which depends on the context) somewhere between the mean and peak rate of the stream. The use of the word “bandwidth” in the phrase “effective bandwidth” has a different meaning than that commonly used by radio engineers. A better name might be “effective data rate.” However, the name “effective bandwidth” is already deeply entrenched in the networking literature, so is used here.

To illustrate the ideas in the simplest setting first, we begin by considering a bufferless communication link, following Hui [20, 21]. The total offered load (measured in bits per second, for example) is given by

$$X = \sum_{j=1}^J \sum_{i=1}^{n_j} X_{ji} \quad (3.39)$$

where J is the number of connection types, n_j is the number of connections of type j , and X_{ji} is the data rate required by the i th connection of type j . Assume that the variables X_{ji} are independent, with the distribution of each depending only on the index j . If the link capacity is C then the probability of overload, $P[X > C]$, can be bounded by Chernoff’s inequality:

$$\log P[X \geq C] \leq \log E[e^{s(X-C)}] = s \left(\sum_{j=1}^J n_j \alpha_j(s) - C \right) \quad (3.40)$$

where $\alpha_j(s)$ is given by

$$\alpha_j(s) = \frac{1}{s} \log E[e^{sX_{ji}}]. \quad (3.41)$$

Thus, for a given value of γ , the quality of service constraint $\log P[X > C] \leq -\gamma$ is satisfied if the vector $n = (n_1, \dots, n_J)$ lies in the region

$$A = \{n \in R_+^J : \min_{s>0} [s \sum_{j=1}^J n_j \alpha_j(s) - C] \leq -\gamma\} = \cup_s A(s) \quad (3.42)$$

where

$$A(s) = \{n \in R_+^J : \sum_{j=1}^J n_j \alpha_j(s) \leq C - \frac{\gamma}{s}\}. \quad (3.43)$$

The *complement* of A relative to R_+^J is convex. Let n^* be on the boundary of A (think of n^* as a “nominal” value of the vector n). A polyhedral subset of A , delineated by a hyperplane tangent to the boundary of A at n^* , is given by $A(s^*)$, where s^* achieves the minimum in (3.42). Thus, any vector $n \in Z_+^J$ satisfying

$$\sum_{j=1}^J n_j \alpha_j(s^*) \leq C - \frac{\gamma}{s^*} \quad (3.44)$$

satisfies the quality of service constraint. Once C , γ , and s^* are fixed, the sufficient condition (3.44) is rather simple. The number $\alpha_j(s^*)$ is the effective bandwidth of a type j connection, and $C - \gamma/s^*$ is the effective capacity. Condition (3.44) is analogous to the condition in classical information theory

which insures that a particular channel is capable of conveying several independent data sources within specified average distortions, namely that the sum of the rate distortion functions evaluated at the targeted distortions should be less than or equal to the channel capacity.

As long as the random variables X_{ji} are not constant, the function α_j is strictly increasing, and ranging from the mean, $E[X_{ji}]$ as $s \rightarrow 0$, to the peak (actually the essential supremum, $\sup\{c : P[X_{ji} > c] > 0\}$), of X_{ji} . Note that the effective bandwidth used depends on the variable s^* . Such dependence is natural, for there is a tradeoff between the degree of statistical multiplexing utilized and the probability of overload, and the choice of the parameter s^* corresponds to selecting a point along that tradeoff curve. Roughly speaking, as the constraint on the overflow probability becomes more severe, a larger value of s^* is appropriate. For example, if γ is very large, then the sets A_s are nonempty only for large s , so that the choice of s^* is also large, meaning that the effective bandwidths will be near the peak values.

The set A defined in (3.42) is only an inner bound to the true acceptance region for the constraint $\log P[X > C] \leq -\gamma$. However, if C and γ both tend to infinity with their ratio fixed, then the true acceptance region, when scaled by dividing by C , converges to A scaled by dividing by C [23]. (The scaled version of A depends only on the ratio of C to γ .) This follows from Cramér's theorem (see [36]), to the effect that Chernoff's bound gives the correct exponent.

So far, only a bufferless link confronted with demand that is constant over all time has been considered. The notion of effective bandwidth can be extended to cover sources of data which vary in time, but which are statistically stationary and mutually independent [3, 37, 7]. Let $X_{ji}[a, b]$ denote the amount of traffic generated by the i th connection of type j during an interval $[a, b]$. We assume that the process X is stationary in time. Set

$$\alpha_j(s, t) = \frac{1}{st} \log E[e^{sX_{ij}[0, t]}]. \quad (3.45)$$

For t fixed, the function α_j is the same as the one-parameter version of α_j considered above, applied to the amount of work generated in an interval of length t . Beginning with the well-known representation of Loynes for the stationary queue length, $Q(0) = \sup_{t \geq 0} X[-t, 0] - tC$, we write

$$\begin{aligned} \log P[Q(0) > B] &= \log P[\sup_{t \geq 0} \{X[-t, 0] - tC\} > B] \end{aligned} \quad (3.46)$$

$$\sim \sup_{t \geq 0} \log P[X[-t, 0] - tC > B] \quad (3.47)$$

$$\sim \sup_{t \geq 0} \min_{s \geq 0} \left[st \sum_{j=1}^J n_j \alpha_j(s, t) - s(B + tC) \right] \quad (3.48)$$

The symbol “ \sim ” used in (3.47) and (3.48) denotes asymptotic equivalence of the logarithms of the quantities on either side of the symbol. The use of this symbol is justified by limit theorems in at least two distinct regimes: (1) the buffer size B tends to infinity with n and C fixed, and (2) the elements of the vector n , the capacity C , and the buffer space B all tend to infinity with the ratios among them fixed. Under either limiting regime, the line (3.47) is justified by the fact that the probability of the union of many rare events (with probabilities tending to zero at various exponential rates) is dominated by the probability of the most probable of those events. The line (3.48), which represents the use of the Chernoff bound as in (3.40), relies on the asymptotic

exactness of the Chernoff bound (Cramér's theorem or more general large deviations principles such as the Gärtner-Ellis theorem. See [36]).

Equations (3.46)-(3.48) suggest that the effective bandwidth to be associated with a connection of type j is $\alpha_j(s^*, t^*)$, where t^* achieves the supremum in (3.48), and s^* achieves the minimum in (3.48) for a nominal value n^* of n . The approximate condition for meeting the quality of service requirement $\log P[Q(0) > B] \leq -\gamma$ for n near n^* is then

$$\sum_{j=1}^J n_j \alpha_j(s^*, t^*) \leq C + \frac{B}{t^*} - \frac{\gamma}{s^* t^*}$$

This region scales linearly in γ if n^* , B and C scale linearly in γ , and asymptotically becomes a tight constraint as $\gamma \rightarrow \infty$. The value t^* is the amount of time that the system behaves in an unusual way to build up the queue length just before the queue length exceeds B . The quantity $C + B/t^* - \gamma/s^*t^*$ is the effective capacity of the link. Following [35], we call t^* the critical time scale. In the first limiting regime, described above, t^* tends to infinity, so the effective bandwidth becomes $\alpha_j(\infty, s^*)$. Use of the Gärtner-Ellis theorem of large deviations theory allows the limit theorems in the first limiting regime to be carried out for a wide class of traffic streams with memory.

The above approximation simplifies considerably in the case that the traffic process is Gaussian. In particular, suppose also that there is only one class of customers (so we drop the index j and let n denote the number of connections) and that for each i the traffic $X_i(0, t]$ is Gaussian with mean λt and variance $V(t)$. The corresponding effective bandwidth function is $\alpha(s, t) = \lambda + sV(t)/2t$. Inserting this into (3.48) and then performing the minimization over s yields that

$$\log P[Q(0) > B] \sim -n \inf_t \frac{((c - \lambda)t + b)^2}{2V(t)} \quad (3.49)$$

where b is the buffer space per connection (so $B = nb$) and c is the capacity per connection ($C = nc$).

Suppose $V(t)/t^{2H}$ converges to a finite constant σ^2 as t tends to infinity, where H , known as the Hurst parameter, typically satisfies $1/2 \leq H < 1$. If $H = 1/2$, we see the process does not exhibit long range dependence. In particular, if X has independent increments (therefore the increments of a Brownian motion with drift λ and diffusion parameter σ^2), then $V(t) = \sigma^2 t$ and moreover (3.49) holds with exact equality.

If $H > 1/2$ (but still $H < 1$) then the critical time scale t^* is still finite. That is, even in the presence of long range dependence, the critical time scale is still finite in the limiting regime of C , B and n tending to infinity with fixed ratios among them [35]. The value of $V(t)$ for t larger than t^* therefore does not influence the approximation.

See [11] and [23] for extensive surveys on effective bandwidth, and [43] for a very extensive bibliographic guide to self-similar traffic models and their use. The paper [4] presents significant bounds and analysis related to notions of equivalent bandwidth with a different terminology. Finally, the paper [22] connects the theory of effective bandwidths to thermodynamics and statistical mechanics.

3.8 Problems

3.1. A queue with customers arriving in pairs

Customers arrive at a single-server queue two at a time. The arrival instants for pairs of customers

are given by a Poisson process with rate λ . The service times of customers are independent, exponentially distributed with parameter μ . Let $N(t)$ denote the number of customers in the system at time t . (a) The process $(N(t) : t \geq 0)$ is a continuous time pure jump Markov process with statespace \mathbb{Z}_+ . Sketch the transition rate diagram of N . (b) See what can be deduced about N by applying the Foster-Lyapunov stability criteria and related moment bounds for the Lyapunov function $V(x) = x$. (c) Repeat part (b) using $V(x) = \frac{x^2}{2}$. (d) Solve for the z transform of the equilibrium distribution and find the equilibrium distribution. Under what condition does it exist (i.e. under what condition is N positive recurrent?) (e) Let $(r_k : k \geq 0)$ and $(d_k : k \geq 0)$ be the equilibrium probabilities defined by

$$\begin{aligned} r_k &= P[\text{there are } k \text{ in the system just before the arrival of a pair of customers}] \\ d_k &= P[\text{there are } k \text{ in the system just after the departure of a typical customer}]. \end{aligned}$$

Express $(r_k : k \geq 0)$ and $(d_k : k \geq 0)$ in terms of $(p_k : k \geq 0)$, λ and μ . In particular, does $r_k = p_k$? or $d_k = p_k$? or $r_k = d_k$?

3.2. Token bucket regulation of a Poisson stream

A token bucket traffic regulation scheme works as follows. The packet stream to be regulated is modeled as a Poisson stream with arrival rate λ . There is a token pool that holds up to B tokens. When a packet arrives, if there is a token in the token pool then the packet instantly passes through the regulator, and one of the tokens is removed from the token pool. If instead the token pool is empty, then the packet is lost. (Notice that packets are never queued.) New tokens are generated periodically, with one time unit between successive generation times. If a token is generated when the token pool is full, the token is lost. (The token pool acts like a "leaky bucket".) (a) Identify an embedded discrete-time, discrete-state Markov process, and describe the one-step transition probabilities of the chain. (b) Express the fraction of packets lost (long term average) in terms of λ , B and π , where π denotes the equilibrium probability vector for your Markov chain. (You do NOT need to find π .) (c) As an approximation, suppose that the times between new token generations are independent, exponentially distributed with common mean 1. Find a fairly simple expression for the loss probability.

3.3. Extremality of constant interarrival times for G/M/1 queues

Consider two G/M/1 queueing systems with common service rate μ . The first has a general interarrival distribution function $A(t)$ with mean $1/\lambda$ such that $\lambda/\mu < 1$. The second has constant interarrival times, all equal to $1/\lambda$. (Thus the interarrival distribution function in the second system is $A_d(t) = 0$ if $t < 1/\lambda$ and $A_d(t) = 1$ otherwise, and the system is called a D/M/1 system). (a) Show that the Laplace transforms are ordered: $A_d^*(s) \leq A^*(s)$ for all $s \geq 0$. (b) Show that the mean number in the system at arrival times (in equilibrium) and the mean waiting time in the system is smaller for the D/M/1 system.

3.4. Propagation of perturbations

Consider a single-server queueing system which is initially empty and for which a total of five customers arrive, at times 1,2,5,13, and 14, respectively. Suppose the amounts of time required to serve the customers are 5,2,4,2,2 time units, respectively. (a) Sketch the unfinished work in the system as a function of time. Indicate the departure times of the five customers, and compute the waiting times in queue of the five customers. (b) Repeat part (a), assuming that the service time

of the second customer is increased to 3 time units. (c) Repeat part (a), assuming the service time of the second customer is increased to 4 time units. (d) Describe briefly in words how in general an increase in service time of one customer effects the waiting times of other customers in the system.

3.5. On priority M/GI/1 queues

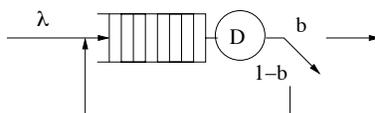
Consider an M/GI/1 system with *preemptive resume* priority service. There are assumed to be two priority classes, with independent arrival streams and service requirements. Class 1 has priority over class 2. Customers of the i th class arrive according to a Poisson process of rate λ_i , and the service times of customers of class i are independent and identically distributed. Use X_i to denote a random variable with the same distribution as the service requirements for class i , for $i = 1$ or $i = 2$. (a) Describe what “preemptive resume priority” means. (b) *Derive* the mean time in the *system*, T_1 and T_2 , for class 1 and class 2, respectively.

3.6. Optimality of the μc rule

Consider an M/G/1 queue with customer classes 1 through K and a nonpreemptive priority service discipline. Let $\mu_i = 1/\bar{X}_i$ where \bar{X}_i denotes the mean service time for type i customers and let c_i , with $c_i > 0$, denote the cost per unit time of holding a class i customer in the queue. For some permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_K)$ of $(1, 2, \dots, K)$, the class σ_1 customers are given the highest priority, the class σ_2 customers are given the next highest priority, and so on. The resulting long run average cost per unit time is $\sum_i \lambda_i c_i W_i(\sigma)$, where $W_i(\sigma)$ is the mean waiting time of class i customers under the priority order σ . (a) Show that an ordering σ minimizes the long-run average cost per unit time over all possible orderings if and only if $\mu_{\sigma_1} c_{\sigma_1} \geq \mu_{\sigma_2} c_{\sigma_2} \geq \dots \geq \mu_{\sigma_K} c_{\sigma_K}$. (Hint: If σ does not satisfy the given condition, then for some i , $\mu_{\sigma_i} c_{\sigma_i} < \mu_{\sigma_{i+1}} c_{\sigma_{i+1}}$. Appeal to the conservation of work equation to argue that by swapping σ_i and σ_{i+1} , an ordering $\hat{\sigma}$ with strictly smaller cost than σ is obtained. You still also need to figure out how to prove both the “if” and “only if” portions of the statement.) (b) Write a brief intuitive explanation for the result of part (a).

3.7. A queue with persistent customers

Consider a queue with feedback as shown, where $\lambda, D > 0$ and $0 \leq b \leq 1$. New arrivals occur according to a Poisson process of rate λ . The service time of a customer for each visit to the queue is the constant D . Upon a service completion, the customer is returned to the queue with probability $1 - b$.



- Under what condition is the system stable? Justify your answer.
- Suppose the service order is FIFO, except that a returning customer is able to bypass all other customers and begin a new service immediately. Denote this by PR, for priority to returning, service order. Express the mean total system time of a customer, from the time it arrives until the time it leaves the server for the last time, in terms of λ, b , and D . (Hint: A geometrically distributed random variable with parameter p has first moment $\frac{1}{p}$ and second moment $\frac{2-p}{p^2}$.)
- If instead the service within the queue were true FIFO, so that returning customers go to the end of the line, would the mean total time in the system be larger, equal, or smaller than for PR service order? Would t

he variance of the total time in the system be larger, equal, or smaller than for PR service order?

3.8. A discrete time M/GI/1 queue

Consider a single server queue in discrete time. Suppose that during each time slot, one customer arrives with probability p , and no customers arrive with probability $1 - p$. The arrival events for different slots are mutually independent. The service times are independent of the arrival times and of each other, and the service time of any given customer is equally likely to be 1,2,3,4,5 or 6 slots. Find W , the mean number of complete slots spent in the queue by a typical customer (don't include time in service). Show your reasoning. Also identify an embedded Markov chain for the system.

3.9. On stochastic ordering of sampled lifetimes

Let f and g be two different probability density functions with support \mathbb{R}_+ , and with the same finite mean, m_1 . Let f_L and g_L denote the corresponding sampled lifetime densities: $f_L(x) = \frac{xf(x)}{m_1}$ and $g_L(x) = \frac{xg(x)}{m_1}$. Show that it is impossible that $f_L \prec g_L$. (We write $f_L \prec g_L$ to denote that a random variable with pdf f_L is stochastically smaller than a random variable with pdf g_L . Note that the mean of f_L does not have to equal the mean of g_L .)

3.10. Effective bandwidth, bufferless link

Consider a bufferless link of fixed capacity $C = 200$ serving connections of 2 types. The data rate of each connection is random, but constant in time. The data rate required by a connection of type 1 is uniformly distributed over $[0,2]$, the data rate required by a connection of type 2 is exponentially distributed with mean 1, and requirements for different connections are mutually independent. (a) Calculate the effective bandwidth functions $\alpha_1(s)$ and $\alpha_2(s)$. (b) Find the largest integer n so that the Chernoff inequality implies that the blocking probability is less than or equal to 0.001 for a nominal load of n connections of each type. Show your work for full credit! (Hint: Write a short computer program for this problem. For a given value of n the Chernoff bound is computed by a minimization over s . Then n can be adjusted by binary search – increasing n if the corresponding overflow probability is too small, and decreasing n if the corresponding overflow probability is too large.) (c) Compute the effective bandwidths $\alpha_1(s^*)$, $\alpha_2(s^*)$, and the effective capacity $C + \frac{\log(0.001)}{s^*}$. Which of the two effective bandwidths is larger? Sketch the corresponding acceptance region $A(s^*)$. Here s^* denotes the optimizing value of s in the Chernoff bound. (Hint: The nominal load point (n,n) should be contained in the region $A(s^*)$. Which of the two effective bandwidths is larger?)

3.11. Effective bandwidth for a buffered link and long range dependent Gaussian traffic

Consider a link with buffer size $B = 100$ and fixed capacity $C = 200$, serving connections of 2 types. The amount of data offered by a connection of type i over an interval of length t is assumed to be Gaussian with mean t and variance $V_i(t) = t^{2H_i}$, where H_i is the Hurst parameter. Assume $H_1 = 0.5$ and $H_2 = 0.9$. (So the class 2 connections have long-range dependence.) (a) Compute the largest value n so that the Chernoff approximation for the overflow probability for n connections of each type (simultaneously) is less than or equal to $\exp(-\gamma) = 0.001$. (Hint: Write a short computer program for this problem. For a given value of C and t the approximation is computed by a minimization over s , which can be done analytically. The maximization over t can be done numerically, yielding the approximate overflow probability for the given value of C . Finally, C can be adjusted by binary search as in the previous problem. Also, compute the effective bandwidths

$\alpha_1(s^*, t^*)$, $\alpha_2(s^*, t^*)$, the critical time t^* , and the effective capacity $C_{eff} = C + \frac{B}{t^*} - \frac{\gamma}{s^* t^*}$. Sketch the corresponding acceptance region $A(s^*, t^*)$. (Hint: The nominal load point (n, n) should be contained in the region $A(s^*, t^*)$. Which of the two effective bandwidths is larger?) (b) Redo problem (a) for overflow probability $\exp(-\gamma) = 0.00001$. and comment on the differences between the answers to parts (a) and (b).

3.12. Extremal equivalent bandwidth for fixed mean and range

Suppose X is a random variable representing the rate of a connection sharing a bufferless link. Suppose the only information known about the probability distribution of X is that $E[X] = 1$ and $P[0 \leq X \leq 2] = 1$. For a given $s > 0$ fixed, what specific choice of distribution of X meeting these constraints maximizes the equivalent bandwidth $\alpha(s)$, where $\alpha(s) = \frac{\ln(E[e^{sX}])}{s}$?

Chapter 4

Multiple Access

4.1 Slotted ALOHA with Finitely Many Stations

Suppose M stations wish to communicate to each other via a shared broadcast channel. Suppose that time is divided into equal length slots such that the length of each slot is the amount of time needed for a station to transmit one packet of information.

Consider a particular station and a particular time slot. If the station does not have a packet at the beginning of the slot, then it receives a new packet in the slot with probability q_a . A new packet (one that arrives in the slot) cannot be transmitted in the slot. If the station already has a packet to transmit at the beginning of the slot, then it transmits the packet with probability q_x in the slot, and the station is not eligible to receive any new packets in the slot. The events of packet arrival and transmission at the different stations are mutually conditionally independent, given the status of the stations at the beginning of the slot.

If no station transmits a packet in the slot, the channel is said to be “idle” during the slot, and if two or more stations transmit during the slot, a “collision” is said to occur. If either the slot is idle or a collision occurs in the slot, then no packet is successfully transmitted. If exactly one station transmits a packet in the slot, then that transmission is called a “success,” and the packet leaves the system. The station making the successful transmission is eligible to receive a new packet in the next time slot, while all other stations with packets carry their packets over to the next slot.

The parameters of the model are thus M, q_a and q_x . Several aspects of the model involve choices that were made largely for simplicity and tractability. For example, we might have allowed a station to continue receiving packets, even after it already had one to send. However, in some systems the successful transmission of the first packet in a queue at a station is the most crucial, because that packet can reserve slots (possibly on another channel) for the packets in queue behind it. Also implicit in the model is that a station can determine by the beginning of the next slot whether its transmission is successful, so that it knows whether to transmit the same packet again, or to switch back to the mode of receiving a new packet. One way to modify the current model to account for a delay of, say, R time slots would be to time-interleave R independent versions of the system. On the other hand, for networks of small physical size, feedback comes so soon that an unsuccessful transmission can be stopped well before the whole packet is transmitted. That is the idea of *collision detect*.

A final aspect of the model we should comment on is that new packets are treated the same way as packets that have already been transmitted. An alternative is to transmit a new packet

with probability one in the first slot after it arrives, and if such transmission is not successful then the packet should be retransmitted with probability q_r in subsequent slots, until it is successfully transmitted. Thus, we have adopted a *delayed first transmission* (DFT) rather than *immediate first transmission* (IFT) model. The qualitative behavior and exact analysis of the two types of models are similar. The IFT model can lead to lower mean delays for lightly loaded systems. On the other hand, the DFT model is less sensitive to variations in the statistics of the arrival process.

Quantities of interest include the mean throughput and mean delay of the system. Since the stations are interchangeable in the model, the process $(N(k) : k \geq 0)$ is a Markov process, where $N(k)$ is the number of stations with packets at the beginning of a slot k . Given $N(k) = n$, the conditional probability of a successful transmission in slot k , $P_S(n, q_x)$, is given by $P_S(n, q_x) = nq_x(1 - q_x)^{n-1}$, and the conditional probability that there are j arrivals is $b(M - n, q_a, j)$, where $b()$ is the binomial mass function

$$b(M - n, q_a, j) = \begin{cases} \binom{M - n}{j} q_a^j (1 - q_a)^{M - n - j} & 0 \leq j \leq M - n \\ 0 & \text{else.} \end{cases}$$

Furthermore, whether there is a successful transmission is conditionally independent of the number of new arrivals, given $N(k) = n$. Thus, the one step transition probabilities of the process can be expressed as follows.

$$P[N(k + 1) = n + i | N(k) = n] = P_S(n, q_x) b(M - n, q_a, i + 1) + (1 - P_S(n, q_x)) b(M - n, q_a, i)$$

The equilibrium probability distribution π of the Markov process $(N(k))$ can be easily computed numerically by matrix iteration. The mean number of packets in the system and mean throughput are given by $\bar{N} = \sum_{n=0}^M n\pi_n$ and $S = \sum_{n=0}^M \pi_n P_S(n, q_x) = (M - \bar{N})q_a$ respectively, and by Little's law the mean delay is \bar{N}/S .

The dynamical behavior of the system can be better understood by considering the drift of the random processes $(N(k))$. The *drift* at instant k is defined to be $E[N(k + 1) - N(k) | N(k)]$. It is a function of $N(k)$, and is equal to the expected change in N over slot k , given $N(k)$. Since the change in N is the number of arrivals minus the number of departures, the drift can be expressed as $d(N(k))$ where $d(n) = (M - n)q_a - P_S(n, q_x)$.

It is convenient to think of the function $d(n)$ as a one-dimensional vector field. For convenience, consider $d(n)$ for real values of n between 0 and M . The zeros of $d(n)$ are called equilibrium points, and an equilibrium point is called stable if $d(n)$ makes a down-crossing at the equilibrium point. We expect that the sample paths of the Markov process spend much of their time near the stable equilibrium points of $(N(k))$. If q_x is sufficiently small for given values of M and q_a , then there is only a single equilibrium point, which is stable. However, for some parameter values there can be three equilibrium points, two of which are stable. One of the stable points occurs for a large value of n , corresponding to large delay and small throughput, whereas the other equilibrium point occurs for a much smaller value of n . The existence of stable equilibrium points at undesirable values of n indicates the following potential problem with the operation of the system: for a long period of time the system may stay near the desirable stable equilibrium point, but due to occasional large stochastic fluctuations it may move to near the undesirable point. A drastic, and undesirable qualitative change in the performance of the stochastic system can occur rather suddenly.

In many dynamical systems, a control mechanism based on feedback can be used to eliminate undesirable stable equilibrium points. That method is investigated for the ALOHA system in the next two sections.

4.2 Slotted ALOHA with Infinitely Many Stations

No Control

Let A_k denote the number of new packets to arrive in slot k . Assume that the random variables ($A_k : k \geq 0$) are independent, identically distributed with $E[A_k] = \lambda$ and $Var(A_k) < \infty$. For example, A_k might have a Poisson distribution, which arises in the limit as the number of stations tends to infinity with the aggregate rate of traffic generation set to λ . Suppose that each station with a packet at the beginning of a slot transmits in the slot with probability q_x . If there are n such stations then the probability of a success, $P_S(n, q_x)$, is given by $P_S(n, q_x) = nq_x(1 - q_x)^{n-1}$. Since $\lim_{n \rightarrow \infty} P_S(n, q_x) = 0$, there is an n_o so large that $P_S(n, q_x) \leq \lambda/2$ for $n \geq n_o$.

We shall argue that this implies that $P[\lim_{k \rightarrow \infty} N(k) = \infty] = 1$. Conditioned on the event $N(k) \geq n_o$, the increment $N(k+1) - N(k)$ is stochastically greater than or equal to¹ a random variable u , where $u = A_1 - B$, and B is a random variable independent of A_1 such that $P[B = 1] = \lambda/2 = 1 - P[B = 0]$. Note that $E[u] = \lambda/2 > 0$. Moreover, given $N(k) \geq n_o$, the future increments of N are all stochastically larger than u until, if ever, the backlog becomes smaller than n_o . Furthermore, there is a positive probability that the backlog will never again be less than n_o , because of the following fact: If u_1, u_2, \dots are independent, identically distributed random variables with $E[u_i] > 0$, then $P[\min_{k \geq 1} u_1 + \dots + u_k \leq -1] < 1$ (otherwise the Strong Law of Large Numbers would be violated). Thus, conditioned on the event $N(k) \geq n_o$, there is a positive probability that the backlog will converge to infinity and never again cross below n_o . Since the backlog must lie above n_o infinitely often, it follows that $P[\lim_{k \rightarrow \infty} N(k) = +\infty] = 1$.

Centralized (Unrealistic) Control

It is easy to show that for n fixed, $P_S(n, q_x)$ is maximized by $q_x = 1/n$, that $P_S(n, 1/n) = (1 + 1/(n-1))^{-(n-1)} > e^{-1}$ and that $\lim_{n \rightarrow \infty} P_S(1, 1/n) = e^{-1}$. We can try to apply this result to a collision access system as follows. Let $N(k)$ denote the number of stations with packets to transmit at the beginning of slot k , and suppose that each such station transmits its packet in the slot with probability $1/N(k)$. Then if $N(k)$ is not zero, the conditional probability of a successful transmission in the slot given $N(k)$ is greater than e^{-1} . Thus the delay in the system is no larger than that for a discrete time M/M/1 bulk-arrival system in which the service times are geometrically distributed with parameter e^{-1} . (The two processes can be constructed on the same probability space so that the M/M/1 bulk system always has at least as many packets in it as the ALOHA system.) The mean delay will thus be finite if $\lambda < e^{-1}$ and the variance of the number of arrivals in each slot is finite. Unfortunately, this retransmission procedure is far from being practical because it requires that all the stations be aware of the number of stations with packets.

Decentralized Control

The stations can implement a practical variation of the centralized control scheme by estimating the number of stations with packets. Suppose, for example, that immediately after slot k the stations

¹“Stochastically greater than or equal to” refers to the notion of stochastic comparison, discussed in Section 3.5.

learn the value of $Z(k)$, where

$$Z(k) = \begin{cases} 0 & \text{if the slot was idle} \\ 1 & \text{if there was a successful transmission in the slot} \\ e & \text{if there was a collision in the slot.} \end{cases}$$

The actual number of packets in the system evolves according to the equation

$$N(k+1) = N(k) - I_{\{Z(k)=1\}} + A(k), \quad (4.1)$$

where $A(k)$ is the number of packets to arrive during slot k . By definition the term $I_{\{Z(k)=1\}}$ is one if $Z(k) = 1$ and zero otherwise, and the stations learn the value of the term because they observe $Z(k)$. On the other hand $A(k)$ is independent of $N(k)$ and all the feedback up to and including $Z(k)$. Thus, a reasonable way for the stations to update an estimate $\hat{N}(k)$ of $N(k)$ is by the following update equations:

$$\hat{N}(k+1) = (\hat{N}(k) - I_{\{Z(k)=1\}} + c(Z(k)))_+ + \lambda, \quad (4.2)$$

where $c = (c(0), c(1), c(e))$ is to be chosen so that the estimator error $D(k)$ defined by $D(k) = \hat{N}(k) - N(k)$ will tend to remain small for all k . Combining the equations for $N(k+1)$ and $\hat{N}(k+1)$, we see that if $\hat{N}(k) > 1 - \min\{c(0), c(1), c(e)\}$, then $D(k+1) = D(k) - (A(k) - \lambda) + c(Z(k))$. Suppose that all stations with a packet to transmit at the beginning of slot k transmit in the slot with probability $\min(1, 1/\hat{N}(k))$. The drift of $(D(k))$ during slot k is given by

$$\begin{aligned} E[D(k+1) - D(k)|N(k), \hat{N}(k)] = \\ c(0)P[Z=0|N(k), \hat{N}(k)] + c(1)P[Z=1|N(k), \hat{N}(k)] + c(e)P[Z=e|N(k), \hat{N}(k)]. \end{aligned}$$

It is not hard to show that if either $N(k)$ or $D(k)$ are sufficiently large then

$$\begin{aligned} P[Z=0|N(k), \hat{N}(k)] &\approx \exp(-G(k)), \\ P[Z=1|N(k), \hat{N}(k)] &\approx G(k) \exp(-G(k)), \\ P[Z=e|N(k), \hat{N}(k)] &\approx 1 - (1 + G(k)) \exp(-G(k)). \end{aligned}$$

where

$$G(k) = \begin{cases} \frac{N(k)}{\hat{N}(k)} & \text{if } \hat{N}(k) \neq 0 \\ 0 & \text{if } \hat{N}(k) = 0. \end{cases}$$

Therefore, the drift of D satisfies

$$E[D(k+1) - D(k)|N(k), \hat{N}(k)] \approx d(G(k)),$$

where

$$d(G) = c(0) \exp(-G) + c(1)G \exp(-G) + c(e)(1 - (1 + G) \exp(-G)).$$

We would like the sign of the drift of $D(k)$ to be opposite the sign of $D(k)$, in order that the error drift towards zero. Now $D(k)$ is positive if and only if $\hat{N}(k)$ is too large (i.e. larger than $N(k)$), which means that $G(k)$ is less than one. Hence we want to choose the constants $(c(0), c(1), c(e))$ so that $d(G) < 0$ when $G < 1$. Similarly we want $d(G) > 0$ when $G > 1$. By continuity this forces $d(1) = 0$. Since we have only one equation (and two other conditions) for the three constants there

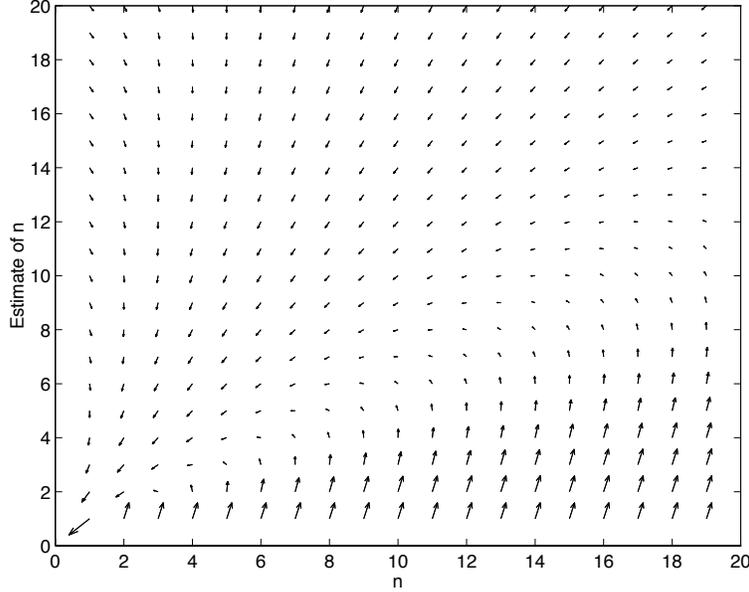


Figure 4.1: Drift field for ALOHA with distributed control.

is some flexibility in choosing the constants. One suitable choice is $c(0) = -(e-2)/(e-1) \approx -0.418$, $c(1) = 0$ and $c(e) = 1/(e-1) \approx 0.582$. To summarize, with this choice of $(c(0), c(1), c(e))$, if the stations use the update formula (4.2) and transmit with probability $1/\hat{N}(k)$ in slot k when they have a packet, then the estimator error $D(k)$ will tend to drift to zero.

As a check on our calculations we can easily do an exact calculation of the drift vector field of the Markov process $(N(t), \hat{N}(t))$, given by

$$d(n, \hat{n}) = E \left[\begin{pmatrix} N(k+1) - N(k) \\ \hat{N}(k+1) - \hat{N}(k) \end{pmatrix} \middle| \begin{pmatrix} N(k) \\ \hat{N}(k) \end{pmatrix} = \begin{pmatrix} n \\ \hat{n} \end{pmatrix} \right].$$

This vector field is plotted in Figure 4.2 for $\lambda = 0.20$ and the above value of c .

The system should perform almost as well as the system with centralized control. The following proposition shows that the system with decentralized control is stable for the same arrival rates λ as the system with centralized control.

Proposition 4.2.1 *Suppose $(c(0), c(1), c(e))$ are chosen so that $d(G) < 0$ if $G < 1$ and $d(G) > 0$ if $G > 1$. Suppose $E[A_k] = \lambda < e^{-1}$, and suppose for some constants a and b , that $P[A_k \geq x] \leq a \exp(-bx)$ for all $x > 0$. Finally, suppose the system begins in the initial state $N(0) = \hat{N}(0) = 0$. Then there exist positive constants A and B with $B < 1$ so that, for all k , $P[N(k) \geq n] \leq AB^n$.*

4.3 Bound Implied by Drift, and Proof of Proposition 4.2.1

Proposition 4.2.1 will be proved in this section by a general method which we call the Lyapunov method of stochastic stability. The idea is to apply bounds implied by drift analysis (explained next) to a one-dimensional process which is a function of the main process under study. The

function mapping the original process into the one dimensional process is called the Lyapunov function. The method offers slick proofs, but, in typical applications, a good understanding of the original process and why it is stable, is needed in order to construct the Lyapunov function.

Bounds Implied by Drift Analysis

Let Y_0, Y_1, \dots be a sequence of random variables. The *drift* at instant k is defined by $E[Y_{k+1} - Y_k | Y_0, Y_1, \dots, Y_k]$. Note that this drift is itself random, being a function of Y_0, \dots, Y_k . For notational convenience, we shall write \mathcal{F}_k instead of Y_0, Y_1, \dots, Y_k when conditioning on these variables. Thus, the drift is $E[Y_{k+1} - Y_k | \mathcal{F}_k]$. If the sequence Y_0, Y_1, \dots is a Markov sequence, then the drift at instant k is a function of Y_k alone.

Let a be a constant with $-\infty \leq a < \infty$, and let ϵ_o, λ and D be strictly positive finite constants. Consider the following conditions on the random sequence:

Condition C.1 The drift at instant k is less than or equal to $-\epsilon_o$ whenever $Y_k \geq a$, for $k \geq 0$. Equivalently,²

$$E[(Y_{k+1} - Y_k + \epsilon_o)I_{\{Y_k \geq a\}} | \mathcal{F}_k] \leq 0 \quad k \geq 0.$$

One might expect that Condition C.1 alone is enough to guarantee that Y_k , at least for large values of k , does not tend to be much larger than the constant a (if a is finite) or that Y_k tends to $-\infty$ in some sense as $k \rightarrow \infty$ (if $a = -\infty$). However, an additional condition is needed (see homework exercise). It is sufficient that the increments of the sequence not be too large. That is the rough idea of Condition C.2:

Condition C.2 There is a random variable Z with $E[\exp(\lambda Z)] = D$ so that

$$P[|Y_{k+1} - Y_k| \geq x | \mathcal{F}_k] \leq P[Z \geq x] \quad x > 0.$$

Let c, η and ρ be constants such that

$$\begin{aligned} c &\geq \frac{E[\exp(\lambda Z)] - (1 + \lambda E[Z])}{\lambda^2} \\ 0 &< \eta \leq \lambda, \\ \eta &< \epsilon_o/c \text{ and,} \\ \rho &= 1 - \epsilon_o \eta + c \eta^2. \end{aligned}$$

Then $\rho < 1$ and $D \geq 1$, and it is not difficult to show that Conditions C.1 and C.2 imply Condition D.1, stated as follows:

Condition D.1 $E[\exp(\eta(Y_{k+1} - Y_k))I_{\{Y_k \geq a\}} | \mathcal{F}_k] \leq \rho$ for $k \geq 0$.

Also, Condition C.2 directly implies condition D.2, stated as follows:

Condition D.2 $E[\exp(\eta(Y_{k+1} - a))I_{\{Y_k < a\}} | \mathcal{F}_k] \leq D$ for $k \geq 0$.

Proposition 4.3.1 *Conditions D.1 and D.2 (and therefore also Conditions C.1 and C.2) imply that*

$$P[Y_k \geq b | Y_0] \leq \rho^k \exp(\eta(Y_0 - b)) + \frac{1 - \rho^k}{1 - \rho} D \exp(-\eta(b - a)). \quad (4.3)$$

²The inequality should be understood to hold except possibly on a set of probability zero, because conditional expectations are only defined up to events of probability zero.

Note that the first term on the right-hand side of (4.3) tends to zero geometrically fast as $k \rightarrow \infty$, and the second term tends to zero exponentially fast as $b \rightarrow \infty$. Note also in the special case $Y_0 \leq a$ that (4.3) yields $P[Y_k \geq b|Y_0] \leq \frac{D \exp(-\eta(b-a))}{1-\rho}$.

Proof. Note that $E[\exp(\eta Y_{k+1})|Y_0] = E[E[\exp(\eta Y_{k+1})|\mathcal{F}_k]|\mathcal{F}_0]$. However

$$\begin{aligned} E[\exp(\eta Y_{k+1})|\mathcal{F}_k] &= E[\exp(\eta(Y_{k+1} - Y_k))I_{\{Y_k \geq a\}}|\mathcal{F}_k] \exp(\eta Y_k) \\ &\quad + E[\exp(\eta(Y_{k+1} - a))I_{\{Y_k < a\}}|\mathcal{F}_k] \exp(\eta a) \\ &\leq \rho \exp(\eta Y_k) + D \exp(\eta a), \end{aligned}$$

so that

$$E[\exp(\eta Y_{k+1})|Y_0] \leq \rho E[\exp(\eta Y_k)|Y_0] + D \exp(\eta a). \quad (4.4)$$

Use of (4.4) and argument by induction on k yields that

$$E[\exp(\eta Y_k)|Y_0] \leq \rho^k \exp(\eta Y_0) + \frac{1 - \rho^k}{1 - \rho} D \exp(\eta a).$$

Combining this inequality with the simple Chernoff bound,

$$P[Y_k \geq b|Y_0] = P[\exp(\eta Y_k) \geq \exp(\eta b)] \leq \exp(-\eta b) E[\exp(\eta Y_k)|Y_0],$$

establishes (4.3). ■

Proof of Proposition 4.2.1

In this section we explain how to prove Proposition 4.2.1 by applying Proposition 4.3.1. The drift of the backlog at instant k , (given the state of the system), is $E[N(k+1) - N(k)|N(k), \hat{N}(k)]$. If this drift were less than some fixed negative constant whenever $N(k)$ is sufficiently large, then Proposition 4.3.1 could be directly applied to prove Proposition 4.2.1. By comparison with the centralized control scheme, we see that the drift is negative if the estimate $\hat{N}(k)$ is close enough to $N(k)$. It is possible that $\hat{N}(k)$ is not close to $N(k)$ (so that $N(k)$ itself doesn't have negative drift), but in that case the estimation error should tend to decrease.

To turn this intuition into a proof, let $Y_k = V(N(k), \hat{N}(k))$ where $V(n, \hat{n}) = n + \phi(|\hat{n} - n|)$ with

$$\phi(u) = \begin{cases} \frac{u^2}{K} & \text{if } 0 \leq u \leq K^2 \\ 2Ku - K^3 & \text{if } u > K^2. \end{cases}$$

for some positive constant K to be specified later. Here V is the Lyapunov function which we use to convert the problem of establishing stability for the two dimensional process $(N(k), \hat{N}(k))$ into the problem of establishing stability for a one dimensional process. Thus, we argue that if K is sufficiently large, then the random sequence (Y_k) satisfies Conditions C.1 and C.2 for some finite constants a, ϵ_o, λ , and η , and conditioning on \mathcal{F}_k means conditioning on $((N(j), \hat{N}(j)) : 0 \leq j \leq k)$. Observe that $Y_k = N(k) + \phi(|\hat{N}(k) - N(k)|)$ so that the drift of Y_k is the sum of the drift of $N(k)$ and the drift of $\phi(|\hat{N}(k) - N(k)|)$. To prove that Condition C.1 holds, note that if Y_k is large, and if $|\hat{N}(k) - N(k)|$ is small, then Y_k inherits negative drift from $N(k)$, and the drift of $\phi(|\hat{N}(k) - N(k)|)$ is small because the function ϕ is nearly flat near zero. On the other hand, if Y_k is large, and if $|\hat{N}(k) - N(k)|$ is also large, then Y_k inherits large negative drift from $\phi(|\hat{N}(k) - N(k)|)$. Thus, in

either case, when Y_k is large it has negative drift, so Condition C.1 is satisfied. Details are left to the reader. Condition C.2 is easily established—a key point is that the derivative of the function ϕ is bounded by $2K$, so

$$|Y_{k+1} - Y_k| \leq (2K + 1)|N(k + 1) - N(k)| + 2K|\hat{N}(k + 1) - \hat{N}(k)| \quad (4.5)$$

Thus, Proposition 4.3.1 can be applied to $(Y_k : k \geq 0)$, and since $N(k) \leq Y_k$, Proposition 4.2.1 follows immediately.

4.4 Probing Algorithms for Multiple Access

The decentralized control strategy described above has obvious deficiencies. For example, if a collision is observed, then until there is a success, it is clear that there are at least two backlogged stations. The update procedure would need a larger state space to take advantage of such information. One could imagine each station computing the conditional *distribution* of the number of backlogged stations, using a Bayesian update formula at each step (in fact this is described in the paper of A. Segall). This would be realizable if the state space is truncated to a finite set of states for the purpose of the estimator. Some approximate equations motivated by this procedure were given by Rivest. However, a key observation is that any given individual station knows not only the feedback information, but it also knows the times that it itself transmitted in the past. This information can serve to distinguish the station from other stations, thereby helping to resolve conflicts. Algorithms called “probing algorithms” or “tree algorithms” take advantage of this additional information.

First the *basic collision resolution algorithm* (CRA) is described. It operates in discrete time with the same $0-1-e$ feedback model considered for controlled ALOHA. Suppose there are initially k stations with packets to transmit. In the first slot, all stations transmit. If $k=0$ or $k=1$, then at the end of the slot it is clear that all packets are successfully sent. If $k=2$ then a collision occurs in the first slot.

After a collision, all transmitters involved flip a fair coin with possible outcomes 0 or 1. Those that obtain a 0 from their coin flip (those “flipping 0” for short) retransmit in the very next slot, and those flipping 1 retransmit in the first slot after the algorithm is done processing the group of stations that flipped a 0. An example involving four stations with packets, A, B, C and D , is pictured in Figure 4.2. Suppose after the initial collision that stations B and C flip 0. Then both B and C transmit in the second slot, causing another collision. If then C flips 0 and B flips 1, then C and B are transmitted successfully in the third and fourth slots respectively. After the fourth slot, the stations with packets A and D realize it is time to transmit again, resulting in a collision in the fifth slot, and so on.

The progress of the algorithm can be traced on a binary tree, as pictured in Figure 4.3. Each node of the tree corresponds to a single slot of the algorithm. Each terminal node of the tree constructed is labeled either 0 or 1, whereas each nonterminal node is labeled e . The algorithm corresponds to a virtual stack, as pictured in Figure 4.4. Packets all initially enter the top level, called level 0, of the stack. All packets at level 0 at the beginning of a slot are transmitted. If there is only one packet at level 0, it is thus successfully transmitted. If there are two or more packets at level 0, then all such packets flip a fair coin, and those that flip 0 stay at level 0 for the next slot, and all those that flip 1 move down to level 1. A packet that is in any other level l at the beginning of a slot is not transmitted in the slot. At the end of the slot the packet moves to level

$l + 1$ if there is a collision in the slot, and moves to level $l - 1$ if there is no collision (i.e. either no transmission or a success) in the slot. For actual implementation of the algorithm, a station with a packet maintains a single counter to keep track of the level of the packet in the virtual stack.

The basic CRA is now completely described. Some comments are in order. First, whenever there is a collision followed by an idle slot, the next slot will be a collision. For example, the outcomes of slots 5 and 6 in the example insure that slot 7 will be a collision. Consequently, the CRA can be “improved” by skipping a step in the algorithm. However, this improvement can lead to instability if the model assumptions are slightly violated—see problem XXX. Another observation is that all packets can sometimes be successfully transmitted a slot or more before the algorithm is complete. In the example, the algorithm requires eleven slots, even though all four packets are transmitted in ten slots. The eleventh slot is necessary in order for all stations listening to the channel outputs to learn that the algorithm is indeed complete.

Let L_n denote the mean number of slots needed for a CRA to process a batch of n packets. Then $L_0 = L_1 = 1$. To find L_n for $n \geq 2$, the outcome of the first slot is to be considered. Then,

$$\begin{aligned} L_n &= 1 + \sum_{k=0}^n \binom{n}{k} 2^{-n} E[\text{additional slots required} | k \text{ stations flip 0 after first slot}] \\ &= 1 + \sum_{k=0}^n \binom{n}{k} 2^{-n} (L_k + L_{n-k}) \\ &= 1 + 2 \sum_{k=0}^n \binom{n}{k} 2^{-n} L_k. \end{aligned}$$

Observing that L_n appears on the right side with a coefficient $2^{-(n-1)}$, we can solve for L_n yielding the recursion:

$$L_n = \frac{1 + 2 \sum_{k=0}^{n-1} \binom{n}{k} 2^{-n} L_k}{1 - 2^{-(n-1)}} \quad (4.6)$$

Let τ denote the (random) completion time given the number of packets has a Poisson distribution with mean μ . The mean of τ can be computed numerically using the expression

$$\bar{\tau} = \sum_{n=0}^{\infty} \frac{\mu^n e^{-\mu} L_n}{n!}.$$

A more direct way is to note that

$$\begin{aligned} \bar{\tau} &= E[\text{number of nodes of tree visited}] \\ &= 1 + 2E[\text{number of nodes corresponding to collisions}] \\ &= 1 + 2 \sum_{k=0}^{\infty} 2^k P[\text{a given node in level } k \text{ of full binary tree corresponds to a collision}] \\ &= 1 + 2 \sum_{k=0}^{\infty} 2^k \{1 - (1 + \mu 2^{-k}) \exp(-\mu 2^{-k})\} \end{aligned}$$

The final sum converges quickly, as can be seen from the fact that $1 - (1 + \epsilon) \exp(-\epsilon) = \epsilon^2/2 + o(\epsilon^2)$ as $\epsilon \rightarrow 0$. Thus, $\bar{\tau}$ is finite for any $\mu > 0$.

4.4.1 Random Access for Streams of Arrivals

So far a method for arranging for transmissions of a fixed group of stations with packets has been described. This collision resolution algorithm can be used as a building block to construct a random access strategy for a continuous stream of arrivals of new packets. In one method, packets just arriving are first transmitted in the first slot after the batch in progress has been handled. This leads to large variations in batch size. Analysis can be based on the observation that the sequence of consecutive batch sizes forms a Markov process. A second method, known as continuous entry, is discussed in Problem xxx. A third method, known as the decoupled window strategy, is discussed in the next subsection.

4.4.2 Delay Analysis of Decoupled Window Random Access Scheme

Under the decoupled window random access scheme, packets are grouped into batches. The k^{th} batch consists of those packets that arrive during the time interval $[(k-1)\Delta, k\Delta)$, where Δ is a parameter of the algorithm. Note that the distribution of the number of packets in a batch is Poisson with mean $\lambda\Delta$. Packets in each batch are processed by a separate run of the basic collision resolution algorithm. The processing time of the k^{th} batch is denoted τ_k . The variables $\tau_k, k \geq 0$ are independent and each has the same distribution as the completion time τ of the basic CRA when the number of packets involved has the Poisson distribution with mean $\mu = \lambda\Delta$. Processing of the k^{th} batch begins in the first slot after the $(k-1)^{\text{th}}$ batch is processed, or in the first slot after time $k\Delta$, whichever is later.

Let U_k denote the waiting time of the k^{th} batch, which is the amount of time from $k\Delta$ until processing for batch k begins. Note that for suitable initial conditions, $U_k = U_k^o + \gamma_k$, where γ_k is uniform on the interval $[0,1]$, and U_k^o is the waiting time of the k^{th} batch in a modified system which does not have time slotting of the transmission process (cf. section on servers with vacations). The sequence $(U_k^o : k \geq 1)$ satisfies the equations:

$$U_{k+1}^o = (U_k^o - \Delta + \tau_k)_+,$$

which is identical to the update equations for the waiting times in a GI/GI/1 queue (more specifically, a D/GI/1 queue), if the τ_k represent the service times in the GI/GI/1 queue, and if the interarrival times are the constant Δ . For $E[U_k]$ to be bounded as $k \rightarrow \infty$ it is necessary and sufficient that $\bar{\tau}$ and $Var(\tau)$ be finite. In fact, by Kingman's first moment bound,

$$E[U_k^o] \leq \frac{\lambda(Var(\delta) + Var(\Delta))}{2(1 - \bar{\tau}/\Delta)} = \frac{\lambda Var(\tau)}{2(1 - \bar{\tau}/\Delta)}$$

if the load $\rho = \bar{\tau}/\Delta$ is less than one. It can be shown that τ has finite variance for any $\Delta > 0$ by modifying either of the two methods used to compute $\bar{\tau}$.

For $E[U_k^o]$ to be bounded it is thus sufficient that $\lambda < \lambda\Delta/\bar{\tau} = \mu/\bar{\tau}$, where $\mu = \lambda\Delta$. For the basic CRA, the maximum throughput is $\max_{\mu} \mu/\bar{\tau} = 0.429$. ZZZ CHOICE OF Δ . It is 0.462 for the modified CRA.

The maximum throughput can be further increased by terminating the processing of a batch whenever the conditional distribution of the number of remaining packets is Poisson distributed. Those packets can be regrouped with packets arriving during a different time interval with length adjusted so that the mean number of packets in the resulting batch is adjusted to an optimal value. This leads to maximum throughput 0.4872. Finally, with this restart idea used, the tree is no longer

symmetric, so further improvement can be gained by adjusting the bias of the coin used, yielding maximum throughput 0.4878.

4.5 Problems

(I haven't assigned these problems recently, and solutions are not provided for them in the back.-BH)

4.1. M/G/1 queue with bulk arrivals

Consider a single server queueing system in which customers arrive in batches. Suppose that batches arrive according to a Poisson process of rate λ , and suppose that $P[\text{a batch has } k \text{ customers}] = g_k$, independently of the time of arrival and of the size of all other batches. Suppose the service times of all customers are independent with distribution function $B(x)$ and mean $1/\mu$, and suppose the sequence of service times is independent of the arrival process. Suppose that the customers are served in FCFS order, and that the customers within a batch are served in some random order that is independent of the service times of the customers. Think of a "typical" customer as one drawn at random from among the first N customers, all choices being equally likely, where N is very large."

// (a) Define the waiting time of a *batch* to be the length of time from when the batch arrives until some customer in the batch begins service. Under what condition is the limiting batch waiting time finite? Under such condition, what is the mean waiting time of a batch in equilibrium? (Hint: reduce to consideration of an M/G/1 system).

(b) What is the distribution of the total number of customers in the same batch as a typical customer? Given that a typical customer is in a batch of size n , what is the conditional distribution of the number, X , of customers in the same batch as the typical customer which are served before the customer itself is served? What is the unconditional probability distribution of X ? What is $E[X]$? Double check your answers for the case $g_1 = g_2 = 0.5$.

(c) What is the mean waiting time in queue for a typical customer?

(d) (Effect of statistical multiplexing). If the arrival rate is doubled to 2λ , if the batch sizes have the same distribution, and if the service times are cut in half, how does the mean waiting time in the queue change? How about the mean waiting time in the system?

(e) (Effect of increasing burstiness). If the arrival rate is halved to $\lambda/2$, if the batch sizes are doubled, and if the service times are kept the same, how does the mean waiting time in the queue change?

4.2. Basic ALOHA

Consider a basic ALOHA system with M stations, each having buffer size one, as discussed in class. (a) Suppose the arrival probability per slot at an empty station is $q_a = 1/(Me)$, and that the transmission probability per slot at an occupied station is $q_x = 1/M$. Perform a drift analysis of the system. How many equilibrium points are there? Assuming M is large and approximating the mean number of busy stations using drift analysis, derive the throughput and the mean delay. (b) Continue to assume that M is large. Also assume that $q_x = 3/M$ and $q_a = a/M$ for some constant a . For what value of a is $T \approx M/2$ (in other words, about the same as for TDMA), where the mean delay T is again predicted by drift analysis. How many equilibrium points are there for this value of a ?

4.3. Basic ALOHA for finite M

Consider a slotted ALOHA system with M stations, each with an infinite buffer. Each station which, at the beginning of a slot, has at least one packet, transmits a packet with probability

q_x . In addition, each station, whether transmitting or not, receives a packet during a slot with probability q_a . Suppose the stations have unbounded buffers. At the end of each slot, all stations learn how many packets were transmitted. If exactly one is transmitted, that packet leaves the system. Otherwise, no packets leave the system. (a) Identify a Markov chain describing the system and give the one-step transition probabilities. (Hints: What is the state space? To gain intuition, it might help to think about the case $M = 2$ first.) (b) What is a necessary and sufficient condition for the system to be stable? (Be as explicit as possible. You should explain your reasoning but you needn't prove your answer.)

4.4. Drift analysis for two interacting Markov queues

Consider a system with two queueing stations. Suppose the arrival processes to the two stations are independent Poisson processes with the rates λ_1 and λ_2 . Suppose the stations are correlated in the following sense. When there are n_1 customers in the first station and n_2 customers in the second station, then the instantaneous service rate in the first station is $\mu n_1 / (n_1 + n_2)$ and the instantaneous service rate in the second station is $\mu n_2 / (n_1 + n_2)$. That is, the effort of a rate μ exponential server is allocated among the two stations in proportion to their numbers of customers. a) Sketch the drift vector field of the Markov chain (N_1, N_2) assuming that $\lambda_1 = 1, \lambda_2 = 2$ and $\mu = 4$. b) Find the mean total number in the system in equilibrium. c) Under what conditions on λ_1, λ_2 and μ is the Markov chain positive recurrent? Can you prove your answer?

4.5. Is negative drift alone enough?

Let X_1, X_2, \dots be independent random variables with $P[X_k = -2k^2] = 1/k^2$ and $P[X_k = 1] = 1 - 1/k^2$. Let $Y_k = X_1 + X_2 + \dots + X_k$. (a) Find $d_k(n) = E[Y_{k+1} - Y_k | Y_1, \dots, Y_{k-1}, Y_k = n]$. What does your answer suggest about the asymptotic behavior of the sequence $(Y_k : k \geq 1)$? (b) Let A_k be the event $\{X_k = -2k^2\}$. Since the sum $P[A_1] + P[A_2] + \dots$ is finite, the Borel-Cantelli lemma implies that $P[A_k \text{ occurs for infinitely many } k] = 0$. What does this fact imply about the asymptotic behavior of the random sequence $(Y_k : k \geq 1)$? (Does the discrepancy between your answers to (a) and (b) contradict any results from class?)

4.6. Controlled ALOHA

Consider a discrete-time controlled ALOHA system. Suppose that just before slot one there are no packets in the system. The number of new packets that arrive during each slot has the Poisson distribution with mean λ , and each new packet arrives at an empty station (infinite user model). The numbers of arrivals during distinct slots are mutually independent. Each station keeps track of a counter. We use $C(k)$ to denote the (common) value of the counters at the beginning of slot k . We assume $C(1) = 0$ and use the following update rule: $C(k) = (C(k) - 1)_+$ if no packets are transmitted in slot k , and $C(k) = C(k) + 1$ if at least one packet is transmitted in slot k . Suppose that any station with a packet at the beginning of slot k transmits it with probability $f(k)$, where $f(k) = 1/C(k)$ if $C(k) \geq 1$ and $f(k) = 1$ otherwise. (a) Identify a Markov chain describing the system and write down the transition probabilities. (b) For what range of λ is the Markov chain positive recurrent? (Hint: When the backlog is large, what does the mean number of packets transmitted per slot drift towards?)

4.7. A controlled ALOHA system

Consider a discrete-time controlled ALOHA system, as discussed in class. Suppose that just before slot one there are no packets in the system. The number of new packets that arrive during each slot has the Poisson distribution with mean λ , and each new packet arrives at an empty station (infinite user model). The numbers of arrivals during distinct slots are mutually independent. Each

station keeps track of an estimate $\hat{N}(k)$ as follows:

$$\hat{N}(k+1) = (\hat{N}(k) - I_{\{Z(k)=1\}} + c(Z_k))_+ + e^{-1}$$

where $(c(0), c(1), c(e)) = (-0.418, 0, .582)$. (If the e^{-1} here were replaced by λ , this would be identical to the estimator considered in class. The estimator here does not require knowledge of λ .) Suppose that any station with a packet at the beginning of slot k transmits it with probability $\min(1/\hat{N}(k), 1)$. (a) Identify a Markov chain describing the system, and write down the transition probabilities. (b) When $\hat{N}(k)$ and $N(k)$ are both large, the difference $G_o\hat{N}(k) - N(k)$ drifts towards zero for some value of G_o . How is the constant G_o determined? (Be as explicit as possible.) (c) Suppose for a given λ that you have computed G_o . Under what condition will the system be stable (using any reasonable definition of stability)? (d) Extra credit: For what range of λ is the system stable?

4.8. An access problem with three stations

Three stations each have a packet and contend for access over a time-slotted channel. Once a station successfully transmits its packet, it remains silent afterwards. Consider two access methods: (i) Each station transmits in each slot with probability $1/2$ until successful. (ii) The stations use the basic unimproved tree algorithm based on fair coin flips. (a) For each of these two methods, compute the expected number of slots required until all packets are successfully transmitted. (b) Suppose a fourth station is monitoring the channel, and that the fourth station does not know that there are initially three stations with packets. Compute the expected number of slots until the fourth station *knows* that all packets have been successfully transmitted.

4.9. Basic tree algorithm with continuous entry

Packets are transmitted in slots of unit length. Suppose packets arrive according to a Poisson process of rate λ packets per slot. Each packet arrives at a distinct station, so that any station has at most one packet. Each station adopts the following strategy:

* A counter value C is set to zero when a new packet is received.

* If a packet has counter value C at the beginning of a slot, then if $C = 0$ the packet is transmitted in the slot. If the transmission is successful the packet leaves the system. If a collision occurs, the station flips a fair coin (value zero or one) and sets C equal to the value.

else if $C \geq 1$ the packet is not transmitted in the slot. If no

–collision occurs in the slot then C is decreased by one. If a (a) Sketch the movement of packets collision occurs then C is increased by one.

in a virtual stack. (b) Suppose n new packets arrive just before slot zero and that there are no other packets in the network at that time. Let B_n denote the expected number of slots until the feedback indicates that there are no packets in the network. Then $B_0 = B_1 = 1$. By conditioning on the outcome during the first slot, find equations for B_2, B_3, \dots . Can you find the least upper bound on the set of λ such that B_n is finite?

4.10. Simple multiple access with four stations

Four stations, labeled 00,01,10,11, share a channel. Some subset of the stations wish to transmit a packet. The stations execute the basic tree algorithm using the bits in their labels to make decisions about when to transmit. Let R_k denote the mean number of slots needed for this conflict resolution algorithm to complete, given that k of the four stations (chosen at random, uniformly) have a packet to transmit. Include the first slot in your count, so that $R_0 = R_1 = 1$. (a) Find R_2, R_3 and R_4 . (b) Compare to the mean time if instead the decisions in the basic tree algorithm

are made on the basis of independent fair coin flips for $1 \leq k \leq 4$. Also consider the comparison for small values of k when there are 2^J stations for J equal to 5 or 6.



Figure 4.2: Time axis for the basic collision resolution algorithm

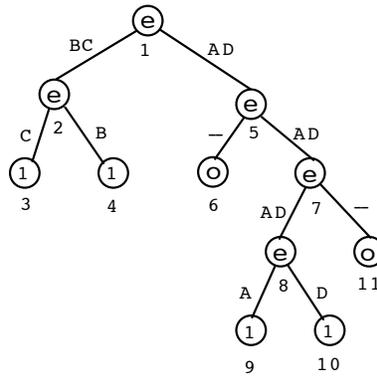


Figure 4.3: Tree for the basic collision resolution algorithm

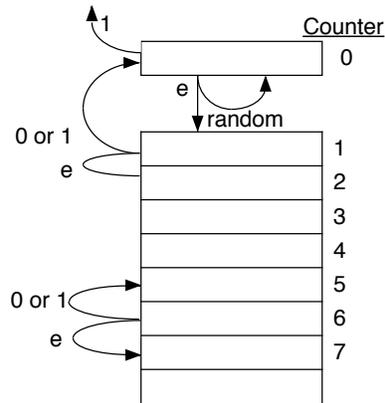


Figure 4.4: Dynamics of the counters for the basic collision resolution algorithm

Chapter 5

Stochastic Network Models

This chapter describes simple stochastic models for two types of networks: circuit switched networks and datagram networks. In circuit switched networks, a given route is considered to be available if there is sufficient capacity at each link along the route to support one more circuit. A performance measure of interest for a circuit switched network is the probability a given route is available. In datagram networks, individual customers move along routes in the network. Performance measures of interest for datagram networks are the transit delay and throughput.

Although arbitrary networks can be quite complicated, the models considered in this chapter have surprisingly simple equilibrium distributions. Furthermore, they suggest useful approximations (such as the reduced load approximation for blocking probability) and they illustrate important phenomena (such as the importance of bottleneck links in heavily loaded store and forward networks).

Before the two types of network models are presented, the theory of reversible Markov processes is introduced. Among other things, the theory of reversible Markov processes suggests rather general techniques for constructing complicated stochastic models which have fairly simple equilibrium distributions.

5.1 Time Reversal of Markov Processes

The *reverse* of a random process $X = (X(t) : t \in \mathbb{R})$ relative to a fixed time τ is $(X(\tau - t) : t \in \mathbb{R})$. Recall that (1.5) gives a characterization of the Markov property which is symmetric in time. Therefore a random process $(X(t) : t \in \mathbb{R})$ is Markov if and only if its reverse is Markov. If X is Markov, the transition probabilities of its reverse process are obtained by

$$P[X(t) = i | X(t+s) = j] = \frac{P[X(t) = i]P[X(t+s) = j | X(t) = i]}{P[X(t+s) = j]}$$

Note that even if X is time-homogeneous, the reverse is not necessarily time-homogeneous.

Suppose $(X(t) : t \in \mathbb{R})$ is a pure-jump, stationary Markov process with countable state space \mathcal{S} and irreducible generator matrix Q . Then the reverse process is also stationary, and its equilibrium distribution π' , transition probabilities $(p'_{ji}(t))$, and generator matrix Q' are given by

$$\pi' = \pi, \quad p'_{ji}(t) = \frac{\pi_i p_{ij}(t)}{\pi_j} \quad t \geq 0 \quad \text{and} \quad q'_{ji} = \frac{\pi_i q_{ij}}{\pi_j}. \quad (5.1)$$

In vector-matrix form, this last relation becomes

$$Q' = \text{diag}(\pi)^{-1} Q^T \text{diag}(\pi)$$

where

$$\text{diag}(\pi) = \begin{pmatrix} \pi_1 & 0 & 0 & \dots \\ 0 & \pi_2 & & \\ & & \ddots & \\ & & & \end{pmatrix}.$$

For example, the transition rate diagram of a stationary Markov process is shown in Fig. 5.1, along with the transition rate diagram of the corresponding reverse process. Note that the process and its reverse have different transition rates for any $\alpha > 0$. However, the sum of the rates leading out of any given state is the same for the forward and reverse time processes. This reflects the fact that the mean holding time for any given state, which is the reciprocal of the sum of the rates out of the state, should be the same for the forward and reverse processes.

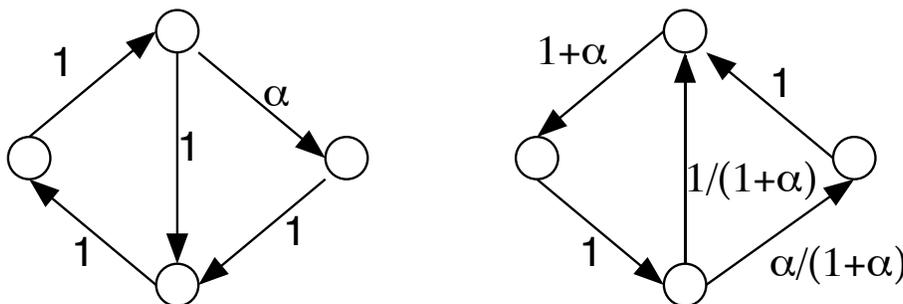


Figure 5.1: Example of forward and reverse transition rate diagrams

The notion of reverse process is useful for verifying that a given distribution is the equilibrium distribution, as indicated in the following proposition. The proof is straightforward and is left to the reader.

Proposition 5.1.1 *Let X be an irreducible, pure-jump Markov process with generator matrix Q , let π be a probability vector and Q' be a generator matrix such that*

$$\pi_i q_{ij} = \pi_j q'_{ji}; \quad \forall i, j \in \mathcal{S}.$$

Then π is the equilibrium distribution of X and Q' is the generator of the reverse process.

A random process is *reversible* if the process and its reverse (relative to a fixed time τ) have the same finite-dimensional distributions, for all τ . Due to the phrase “for all τ ” in this definition, a reversible process is necessarily stationary.

The pure-jump stationary Markov process X is *reversible* if and only if $Q = Q'$, or equivalently, if and only if the *detailed balance* equations hold:

$$\pi_i q_{ij} = \pi_j q_{ji} \quad \forall i, j. \tag{5.2}$$

In comparison, the global balance equations are $\pi Q = 0$, or equivalently

$$\sum_{i:i \neq j} \pi_i q_{ij} = \sum_{i:i \neq j} \pi_j q_{ji} \quad \forall j.$$

Note that it is easy to construct reversible Markov processes as follows. First select a state space \mathcal{S} and an arbitrary probability distribution π on \mathcal{S} . Then for each pair of states i and j , simply select any values for q_{ij} and q_{ji} such that (5.2) holds. Also, if \mathcal{S} is infinite, care should be taken that the sum of the rates out of any state is finite.

Reversible Markov processes also arise naturally. Consider for example an irreducible, stationary birth-death processes, and let $n \geq 0$. The only way the process can move between the subsets $\{0, \dots, n\}$ and $\{n+1, n+2, \dots\}$ is to make a jump from n to $n+1$ or from $n+1$ to n . Thus, $\pi_n \lambda_{n+1} = \pi_{n+1} \mu_{n+1}$. This relation yields that the detailed balance equations (5.2) hold, so that the process is reversible.

More generally, let X be an irreducible stationary pure-jump Markov process, and let $G = (V, E)$ denote the undirected graph with set of vertices \mathcal{S} and set of edges $E = \{[i, j] : i \neq j, \text{ and } q_{ij} > 0 \text{ or } q_{ji} > 0\}$. The graph G is called a tree graph if it is connected (which it must be because X is assumed to be irreducible) and cycle-free. In case \mathcal{S} is finite this is equivalent to assuming that G is connected and has $|\mathcal{S}| - 1$ edges, or equivalently, G is connected but the deletion of any edge from G divides G into two disjoint subgraphs. The same reasoning used for birth-death processes (which are special cases) shows that any stationary Markov process so associated with a tree graph is reversible.

A useful way to obtain reversible processes with larger state spaces is to consider collections of independent reversible processes. That is, if $X(t) = (X_1(t), \dots, X_n(t))$ for all t , where the components $X_i(t)$ are mutually independent reversible processes, then X itself is a reversible process.

If X is an irreducible reversible pure-jump Markov process with generator matrix Q , then a new such process can be obtained as follows. Fix any two distinct states i_o and j_o and let $\alpha > 0$. The new generator matrix is the same as Q , except that $q_{i_o j_o}$ and $q_{j_o i_o}$ are replaced by $\alpha q_{i_o j_o}$ and $\alpha q_{j_o i_o}$ respectively, and the new values of $q_{i_o i_o}$ and $q_{j_o j_o}$ are chosen so that the new matrix also has row sums equal to zero. We can as well take $\alpha = 0$, although doing so might destroy the irreducibility of the resulting Q matrix, so that more than one modified process can arise, depending on the equilibrium distribution.

Taking this last idea a bit further, given a subset A of \mathcal{S} we can obtain a generator for a Markov process with state space A by simply restricting Q to the set $A \times A$, and suitably readjusting the diagonal of Q . The resulting process is called the *truncation* of X to A .

As we shall see in the next section, the above ideas of considering collections of processes (to build up size) and truncation (to introduce nontrivial dependence among the constituent processes) is a powerful way to construct complex reversible models in a natural way.

Exercise: Consider the above modification if $\alpha \rightarrow \infty$. In some sense the states i_o and j_o become aggregated in this limit.

5.2 Circuit Switched Networks

Consider a circuit switched network with R routes and M links. A route is simply a subset of links. Usually such links form a path, but that is not necessary in what follows. An example network with links $\{1, 2, 3, 4\}$ and four routes $\{1, 2, 3\}, \{1\}, \{2, 3\}, \{2, 4\}$, is shown in Figure 1.2. Suppose that calls

arrive for route r according to a Poisson process at rate ν_r , and that call durations are independent and exponentially distributed with mean one.

Let $X_r(t)$ denote the number of calls in progress on route r at time t , and let $Z_l(t)$ denote the resulting load on link l , defined by

$$Z_l(t) = \sum_r A_{lr} X_r(t),$$

where A is the link-route incidence matrix defined by $A_{lr} = 1$ if $l \in r$ and $A_{lr} = 0$ otherwise. In matrix notation, $Z(t) = AX(t)$ for all t .

Then $X = ((X_1(t), \dots, X_R(t))^T : t \geq 0)$ is a Markov process with state space \mathbf{Z}_+^R . If there are no capacity constraints imposed on the links, then the components of X are independent, and each behaves as an M/M/ ∞ system. An M/M/ ∞ system is of course a special type of birth-death process, and is thus reversible. Therefore X itself is reversible.

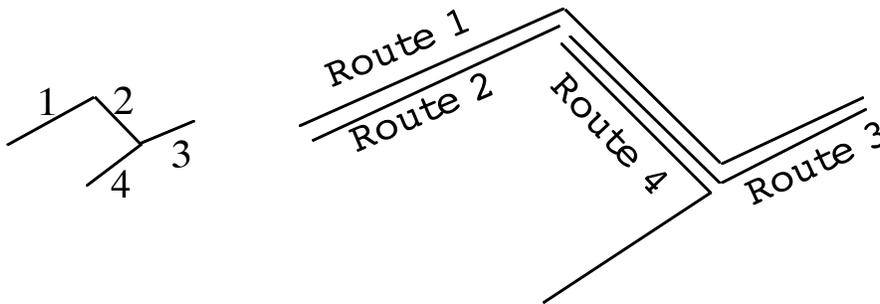


Figure 5.2: A sample circuit switched network, with routes indicated.

Let e_i denote the column vector with a 1 in the i th position and 0's in all other positions. The Q matrix is given by (only nonzero, off-diagonal entries are displayed):

$$\begin{aligned} q(x, x + e_r) &= \nu_r & \text{if } x \in \mathbf{Z}_+^R \\ q(x, x - e_r) &= x_r & \text{if } x - e_r \in \mathbf{Z}_+^R \end{aligned}$$

and the equilibrium distribution π is given by

$$\pi(x) = \prod_r \frac{\nu_r^{x_r} \exp(-\nu_r)}{x_r!}$$

We remark that the process Z is not necessarily Markov. Consider the example in Figure 5.2. If it is known that $Z(t) = (1, 1, 1, 0)$, then it is not determined whether there is a single call, on route 1, or two calls, one on route 2 and one on route 3. The history of Z before t can distinguish between these two possibilities, giving additional information about the future of Z .

If capacity constraints are imposed, so that link l can support at most C_l calls, then the state space is truncated down to $\mathcal{S}_C = \{x \in \mathbf{Z}_+^R : Ax \leq C\}$, where C is the column vector of capacities.

The truncated system is also reversible, and the equilibrium distribution¹ is

$$\pi^C(x) = \frac{1}{Z(C)} \prod_r \frac{\nu_r^{x_r} \exp(-\nu_r)}{x_r!}$$

where the constant $Z(C)$ is chosen to make π^C a probability distribution.

In the remainder of this section we examine the probability of blocking for calls offered for a given route. If the network has only a single link, the total number of calls in progress behaves like an $M/M/C_1/C_1$ system, so that

$$P[k \text{ calls in progress}] = \frac{\lambda_1^k/k!}{\sum_{j=0}^{C_1} \lambda_1^j/j!},$$

where λ_1 is the sum of the arrival rates. In particular, the blocking probability at the link is given by $B_1 = P[C_1 \text{ calls in progress}] = E(\lambda_1, C_1)$, where E denotes the Erlang blocking formula.

In general, a link l is full with probability $B_l = 1 - (Z(C - e_l)/Z(C))$. For large networks, this is a difficult quantity to compute, which leads us to consider bounds and approximations. First, a bound is presented. Define λ_l for a link l as the sum of ν_r , over all routes r that include link l . In matrix notation, this is $\lambda = A\nu$. The following conservative bound on the link blocking probability is intuitively clear:

$$B_l \leq E(\lambda_l, C_l) \tag{5.3}$$

A proof of (5.3) goes as follows. Fix a link l . The right hand side, $E(\lambda_l, C_l)$, is the same as the blocking probability in the original network if all link capacities except C_l are changed to $+\infty$. It is easy to construct the original network process and the modified process on the same probability space so that, with probability one, the number of calls in progress through line l for the original process is less than or equal to the number of calls in progress for the modified process. Indeed, the sequences of offered arrivals should be the same, and departures of routes currently using link l should be governed by C_l independent Poisson clocks. When there are $k \geq 1$ customers in a system (either one) at time $t-$ then there is a departure at time t if one of the first k Poisson clocks rings. In particular, the fraction of time link l is full is smaller for the original process. Since time-averages equal statistical averages for positive recurrent Markov processes, (5.3) follows. What is less obvious (and more difficult to prove) is the remarkable lower bound of W. Whitt (“Blocking When Service is Required from Several Service Facilities,” *AT&T Technical Journal*, vol. 64, pp. 1807-1856, 1985)

$$P[\text{route } r \text{ free}] \geq \prod_{l \in r} (1 - E(\lambda_l, C_l)) \tag{5.4}$$

The bounds (5.3) and (5.4) are often accurate approximations when blocking probabilities are very small (which is, of course, often the case of interest). For moderately loaded networks, the following *reduced load approximation* is often quite accurate:

$$\hat{B}_l = E(\hat{\lambda}_l, C_l), \tag{5.5}$$

¹Amazingly enough, the same equilibrium distribution results if the holding times of calls for each route r are independent and identically distributed (not necessarily exponentially distributed) random variables of unit mean. This is an example of an “invariance” result in queueing theory.

where $\hat{\lambda}_l$ is an estimate of the effective traffic offered at link l :

$$\hat{\lambda}_l = \sum_{r:l \in r} \left\{ \nu_r \prod_{l':l' \in r, l' \neq l} (1 - \hat{B}_{l'}) \right\} \quad (5.6)$$

The product term in (5.6) approximately accounts for the thinning of the traffic offered at link l due to blocking at other links. Taken together, (5.5) and (5.6) yield $2M$ equations for the $2M$ variables $(\hat{B}_l, \hat{\lambda}_l : 1 \leq l \leq M)$. Moreover, the $\hat{\lambda}$'s are determined by the \hat{B} 's, and the \hat{B} 's are determined by the $\hat{\lambda}$'s. The equations can thus be thought of as fixed point equation for the \hat{B} 's (or for the $\hat{\lambda}$'s). They can be numerically solved using iterated substitution or dampened iterated substitution. It can be shown that there is a unique fixed point (F.P. Kelly, "Blocking Probabilities in Large Circuit Switched Networks," *Advances in Applied Probability*, 18, 473-505, 1986). The acceptance probability for a route r can then be approximated as

$$P[\text{route } r \text{ free}] \approx \prod_{l:l \in r} (1 - \hat{B}_l). \quad (5.7)$$

Theory and numerical experience have shown the fixed point equations to be rather accurate for large networks with large capacity constraints and a diverse collection of routes.

5.3 Markov Queueing Networks (in equilibrium)

Networks of queueing stations (we use the word "station" instead of "system" here to better distinguish the elements from the total network) are examined in this section. A simple way to construct such a network is to arrange a sequence of stations in series. Such networks are considered first. Then two more general types of networks are considered, which differ in the routing mechanisms used. In the first, each customer wanders through the network according to a Markov process. In the second model, each customer follows a deterministic route, which depends on the type of the customer. If one views the type of a new customer as being random, one can see that the second routing mechanism is essentially more general than the first. For simplicity, we restrict our attention to Poisson arrival processes and Markov type service mechanisms. More general networks, based on the notion of quasi-reversibility discussed below, are described in Walrand, *An Introduction to Queueing Networks*, Prentice Hall, 1988.

5.3.1 Markov server stations in series

We first consider a single station, paying particular attention to the departure process. A "Markov server" with state dependent rate, denoted as an M_S type server, with parameters $\mu(1), \mu(2), \dots$, is an exponential server with instantaneous rate $\mu(n)$ when there are n customers in the station. That is, given there are n customers in the station at time t , the probability of a service completion during the interval $[t, t + h]$ is $\mu(n)h + o(h)$. If the arrival process is Poisson with rate λ , the state of the resulting number-in-station process is a birth-death Markov process of the type considered in Section 1.12, such as an $M/M/m$ station with $1 \leq m \leq \infty$. The equilibrium distribution exists and is given by:

$$\pi(n) = \frac{\lambda^n}{S_1 \prod_{l=1}^n \mu(l)}, \quad \text{if } S_1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{l=1}^n \mu(l)} < +\infty.$$

Suppose the state process $(N(t))$ is stationary. Since it is a birth-death process it is *time-reversible*. Conclude that the following properties hold. The first property, which implies the second, is known as *quasi-reversibility*.

- The departure process before time t is independent of $N(t)$.
- The departure process is Poisson, with rate λ .

Next, consider a network of M_S stations in series, with the arrival process at the first station being a Poisson process with rate λ . The entire network is described by a Markov process with state space $\mathcal{S} = \{(n_1, n_2, \dots, n_J) : n_i \in Z_+\} = Z_+^J$. Consider a stationary version of the network state process (which exists if the station is positive recurrent.) Also, fix a time t_o . Since there is no feedback effect in the network, the first station in the network behaves like an isolated M_S server station with Poisson arrivals. Therefore, the departure process from the first station is a Poisson process. In turn, that means that the second station in the network (and by induction, all other stations in the network) behave as an isolated M_S server station with Poisson arrivals of rate λ .

Of course the individual stations are stochastically coupled, because, for example, a departure instant for the first station is also an arrival instant for the second station. However, remarkably enough, for a fixed time t_o , the components $N_1(t_o), \dots, N_J(t_o)$ are independent. One explanation for this is the following. By the first quasi-reversibility property for the first station, $N_1(t_o)$ is independent of departures from station 1 up to time t_o . But those arrivals, together with the service times at the other stations, determine the states of all the other stations. Therefore, $N_1(t_o)$ is independent of $(N_2(t_o), \dots, N_J(t_o))$. Similarly, $N_2(t_o)$ is independent of the past departures from the second station, so that $N_2(t_o)$ is independent of $(N_3(t_o), \dots, N_J(t_o))$. Continuing by induction yields that $N_1(t_o), \dots, N_J(t_o)$ are indeed independent.

Therefore, the equilibrium distribution for the network is given by

$$\pi(n_1, n_2, \dots, n_J) = \prod_{j=1}^J p_j(n_j) \text{ where } p_j(n_j) = \frac{b_j \lambda^{n_j}}{\prod_{l=1}^{n_j} \mu_j(l)}$$

where b_j is a normalizing constant for each j .

Another approach for proving this result is to check that $\pi Q = 0$. That approach is taken in a more general setting in the next subsection. An advantage of the argument just given is that it can be modified to show that the waiting times of a typical customer at the J stages of the network are mutually independent, with each waiting time having the distribution of a single M_S server station with Poisson arrivals in equilibrium.

5.3.2 Simple networks of M_S stations

Consider next a network of J stations, each with M_S type servers. Let $\mu_j(n)$ denote the service rate at station j given n customers are at the station. Customers are assumed to follow routes according to a discrete time Markov process with one step transition probability matrix $R = (r_{ij})_{1 \leq i, j \leq J}$. Thus, given that a customer just left station i , it visits station j next with probability r_{ij} . We distinguish two types of networks, open and closed, as follows.

An *open network* accepts customers from outside the network. Such arrivals are called *exogenous* arrivals. The stream of exogenous arrivals at any given station i is assumed to be a Poisson stream with rate γ_i , and the arrival processes for distinct stations are assumed to be independent. To

avoid an infinite buildup of customers, we assume that R is such that starting from any station in the network, it is possible to eventually leave. In particular, at least some of the row sums of the matrix R are less than one. The vector of exogenous arrival rates $\gamma = (\gamma_1, \dots, \gamma_J)$ and R determine the vector of total arrival rates $\lambda = (\lambda_1, \dots, \lambda_J)$ by the equations

$$\lambda_i = \gamma_i + \sum_j \lambda_j r_{ji} \quad \forall i, \quad (5.8)$$

which in vector-matrix notation can be written $\lambda = \lambda R + \gamma$.

A *closed network* contains a fixed set of customers. There are no exogenous arrivals, nor departures from the network. The rows of R all have sum one. A possibly unnormalized vector of arrival rates $\lambda = (\lambda_1, \dots, \lambda_J)$ satisfies

$$\lambda_i = \sum_j \lambda_j r_{ji} \quad \forall i \quad (5.9)$$

or in vector form, $\lambda = \lambda R$. These are the same equations as satisfied by the equilibrium distribution for a finite state Markov process with one step transition probability matrix R . Assuming that R is irreducible (as we do), the vector λ is uniquely determined by these equations up to a constant multiple.

If for each j the service rate $\mu_j(n)$ is the same for all $n \geq 1$ (so that the service times at station j are exponentially distributed with some parameter μ_j), then the network model of this section is known as a *Jackson network*.

Proposition 5.3.1 *The equilibrium distribution of the network is given by*

$$\pi(n_1, n_2, \dots, n_J) = G^{-1} \prod_{i=1}^J p_i(n_i)$$

where

$$p_i(n_i) = \frac{\lambda_i^{n_i}}{\prod_{l=1}^{n_i} \mu_i(l)}$$

and

$$G = \sum_n \left(\prod_{i=1}^J p_i(n_i) \right).$$

where the sum is over all possible states $n = (n_1, \dots, n_J)$. (For closed networks, write $G(K)$ instead of G to emphasize the dependence of G on K).

Proof. In order to treat both open and closed networks at the same time, if the network is open we convert it into a closed network by adding a new station which represents the “outside” of the original open network. That is, if the network is open, add a station called station 0 (zero) which always has infinitely many customers (so $n_0 \equiv +\infty$). Let the service rate of station 0 be $\mu_0(n_0) \equiv \gamma_{tot} = \sum_{i=1}^J \gamma_i$ and extend the routing matrix by setting

$$r_{0i} = \frac{\gamma_i}{\gamma_{tot}}, \quad r_{00} = 0, \quad \text{and} \quad r_{i0} = 1 - \sum_{j=1}^J R_{ij}.$$

The extended routing matrix R has row sums equal to zero, and the vector λ satisfying (5.8) can be extended to a vector satisfying (5.9) for the extended network by setting $\lambda_0 = \gamma_{tot}$. Then the network can be considered to be closed (albeit with infinitely many customers) so the remainder of the proof is given only for closed networks.

For state $n = (n_1, n_2, \dots, n_J)$ let $T_{ij}n$ denote the new state

$$T_{ij}n = (n_1, n_2, \dots, n_i - 1, \dots, n_j + 1, \dots, n_J) \quad 0 \leq i, j \leq J.$$

The transition rate matrix for the Markov process describing the whole network is defined by (only terms for $n \neq n'$ need be given):

$$Q(n, n') = \begin{cases} \mu_i(n_i)r_{ij} & \text{if } n' = T_{ij}n \text{ some } i, j \quad i \neq j \\ 0 & \text{if } n' \neq T_{ij}n \text{ for any } i, j \quad \text{and } n \neq n' \end{cases}$$

We need to verify the balance equations

$$\sum_{n': n' \neq n} \pi(n')Q(n', n) = \sum_{n: n \neq n'} \pi(n)Q(n, n') \text{ for all } n.$$

Therefore it suffices to note that the stronger statement holds for each i such that $n_i \geq 1$:

$$\sum_{j: j \neq i} \pi(T_{ij}n)Q(T_{ij}n, n) = \sum_{j: j \neq i} \pi(n)Q(n, T_{ij}n), \quad \forall n, \forall i$$

which is equivalent to

$$\sum_{j: j \neq i} \left[\pi(n) \left(\frac{\lambda_j}{\mu_j(n_j + 1)} \right) \left(\frac{\lambda_i}{\mu_i(n_i)} \right)^{-1} \right] \mu_j(n_j + 1)r_{ji} = \pi(n)\mu_i(n_i).$$

which is true because it reduces to (5.8) with $\gamma_i \equiv 0$. ■

TO BE ADDED NOTES HERE: Evaluation of mean delay for open networks, and evaluation of throughput/cycle time for closed networks and investigation for varying number of customers, application to head of line blocking in packet switches.

5.3.3 A multitype network of M_S stations with more general routing

We assume the network has I types of customers. It is assumed that new type i customers are generated according to a Poisson process with rate $\nu(i)$ and that each type i customer has a fixed route, equal to the sequence of stations $(r(i, 1), r(i, 2), \dots, r(i, s(i)))$.

We assume that $\mu_j(n_j)$ is the departure rate of station j if there are n_j customers there, no matter what type of customers are in the station, for $(1 \leq j \leq J)$. For simplicity, we assume that customers at each station are served in first-come, first-served (FCFS) order. (Results exist for many other orderings, but the service order does play a big role in the model.)

The entire network can be modeled as a pure jump Markov process. To obtain a sufficiently large state to obtain the Markov property, it is necessary to keep track not only of the number of customers n_j at each station j , but also the type $t_j(l)$ of the l^{th} customer in station j for $0 \leq j \leq n_j$. Moreover, since we allow a customer to visit a given station more than once, in order to know the

future routing we must also keep track of $s_j(l)$, which is the stage that the l^{th} customer in station j is along its route. For example, if $s_j(l) = 3$ then the l^{th} customer in station j is currently at the third station (which happens to be station j) along its route. A generic state of the network can thus be written as $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_J)$, where for each j , \mathbf{c}_j is the state of station j , written as

$$\mathbf{c}_j = (c_j(1), c_j(2), \dots, c_j(n_j))$$

where $c_j(l) = (t_j(l), s_j(l))$ is called the *class* of the l^{th} customer in station j .

The long term total arrival rate at station j (assuming the network is positive recurrent) is given by

$$\lambda_j = \sum_{\text{types } i} \nu(i) \times \{ \text{number of times route } i \text{ passes through station } j \}$$

Define a probability distribution for the state of station j by

$$\pi_j(\mathbf{c}_j) = b_j \left[\frac{\lambda_j^{n_j}}{\prod_{l=1}^{n_j} \mu_j(l)} \right] \prod_{l=1}^{n_j} \frac{\nu(t_j(l))}{\lambda_j} = b_j \prod_{l=1}^{n_j} \frac{\nu(t_j(l))}{\nu_j(l)}.$$

Under this distribution, the distribution of the total number in station j is the same as in the simpler network of the previous subsection. Furthermore, given n_j , the classes attached to the customers in the n_j positions in station j are independent and proportional to the fraction of arrivals of various classes.

Proposition 5.3.2 *The equilibrium distribution for the network is*

$$\pi(\mathbf{c}) = \prod_{j=1}^J \pi_j(\mathbf{c}_j).$$

Proof. Given a state \mathbf{c} of the network, let $T^i \mathbf{c}$ (respectively $T_j \mathbf{c}$) denote the new state caused by an arrival of an exogenous customer of type i (respectively caused by a departure at station j , which is well defined only if $n_j \geq 1$). Then, the transition rates for the network (given only for $\mathbf{c} \neq \mathbf{c}'$) are

$$Q(\mathbf{c}, \mathbf{c}') = \begin{cases} \mu_j(n_j) & \text{if } \mathbf{c}' = T_j \mathbf{c} \\ \nu(i) & \text{if } \mathbf{c}' = T^i \mathbf{c} \\ 0 & \text{if } \mathbf{c}' \text{ is not of the form } T^i \mathbf{c} \text{ or } T_j \mathbf{c} \text{ (and } \mathbf{c} \neq \mathbf{c}'). \end{cases}$$

Now, some thought (or hindsight!) allows us to make the following guess for transition rates of time-reversed network:

$$Q'(\mathbf{c}', \mathbf{c}) = \begin{cases} \nu(i) & \text{if } \mathbf{c}' = T_j \mathbf{c} \text{ and } s_j(1) = S(t_j(1)), \text{ where } i = t_j(1) \\ \mu_k(n_k + 1) & \text{if } \mathbf{c}' = T_j \mathbf{c} \text{ and } s_j(1) < S(t_j(1)), \text{ where } k = r(t_j(1), s_j(1) + 1) \\ \nu_k(n_k + 1) & \text{if } \mathbf{c}' = T^j \mathbf{c} \text{ where } k = r(j, 1). \end{cases}$$

To complete the proof we need only verify that $\pi(\mathbf{c})Q(\mathbf{c}, \mathbf{c}') = \pi(\mathbf{c}')Q'(\mathbf{c}', \mathbf{c})$ for all states \mathbf{c}, \mathbf{c}' (this is left to the reader) and apply Proposition 5.1.1. ■

5.4 Problems

5.1. Time reversal of a simple continuous time Markov process

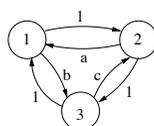
Let $(X(t) : t \geq 0)$ be a stationary time-homogeneous, pure-jump Markov process with state space $\{1, 2, 3, 4\}$ and Q matrix

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 - \alpha & 1 & \alpha \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

where $\alpha > 0$. Find the Q matrix of the time-reversed process. Sketch the rate transition diagram of both X and the time-reversed process.

5.2. Reversibility of a three state Markov process

Consider the continuous-time Markov process with the transition rate diagram shown. Identify all possible values of $(a, b, c) \in \mathbb{R}_+^3$ which make the process reversible.



5.3. A link with heterogeneous call requirements

Let $M = K \times L$, where K and L are strictly positive integers, and let $\lambda_S, \lambda_F \geq 0$. A bank of M channels (i.e. servers) is shared between slow and fast connections. A slow connection requires the use of one channel and a fast connection requires the use of L channels. Connection requests of each type arrive according to independent Poisson processes, with rates λ_S and λ_F respectively. If there are enough channels available to handle a connection request, the connection is accepted, service begins immediately, and the service time is exponentially distributed with mean one. Otherwise, the request is blocked and cleared from the system.

- Sketch a transition rate diagram for a continuous-time Markov model of the system.
- Explain why there is a simple expression for the equilibrium probability distribution, and find it. (You don't need to find an explicit expression for the normalizing constant.)
- Express B_S , the blocking probability for slow requests, and B_F , the blocking probability for fast requests, in terms of the equilibrium distribution.

5.4. Time reversibility of an M/GI/1 processor sharing queue

A processor sharing server devotes a fraction $1/n$ of its effort to each customer in the queue whenever there are n customers in the queue. Thus, if the service requirement of a customer arriving at time s is x , then the customer departs at the first time s' such that $\int_s^{s'} 1/N(t) dt = x$, where $N(t)$ denotes the number of customers in the system at time t . It turns out that if the queue is stable then N is time-reversible, and its equilibrium distribution depends only on the load ρ . In particular, the departure process is Poisson. This result is verified in this problem for a certain class of distributions of the service requirement X of a single customer. (a) Is N Markovian? If not, explain what additional information would suffice to summarize the state of the system at a time t so that, given the state, the past of the system is independent of the future.

- To arrive at a Markov model, restrict attention to the case that the service requirement X of a customer is the sum of k exponentially distributed random variables, with parameters μ_i for

$1 \leq i \leq k$. Note that $E[X] = 1/\mu_1 + \dots + 1/\mu_k$ and $\rho = \lambda E[X]$. Assume $\rho < 1$. Equivalently, X is the time it takes a pure birth Markov process to move from state 1 to state $k+1$ if the birth rate in state i is μ_i . Thus, each customer moves through k stages of service. Given there are n customers in the system, a customer in stage i of service completes that stage of service at rate μ_i/n . Let $N_i(t)$ denote the number of customers in the system that are in stage i of service at time t . Let $\Theta(t) = (N_1(t), \dots, N_k(t))$. Then Θ is a Markov process. Describe its transition rate matrix Q .

(c) The process Θ is not reversible. Show that the equilibrium distribution of Θ is given by $\pi(n_1, \dots, n_k) = (1 - \rho)\rho^n \binom{n}{n_1, \dots, n_k} a_1^{n_1} \dots a_k^{n_k}$, where $n = n_1 + \dots + n_k$ and $a_i = (1/\mu_i)/(1/\mu_1 + \dots + 1/\mu_k)$. At the same time, show that the time reverse of Θ has a structure similar to Θ , except that instead of going through the stages of service in the order $1, 2, \dots, k$, the customers go through the stages of service in the order $k, k-1, \dots, 1$, with the service requirement for stage i remaining exponentially distributed with parameter μ_i . (It follows that then number of customers in the system in equilibrium has distribution $P[N(t) = n] = (1 - \rho)\rho^n$.) (d) Explain why $N(t)$ is reversible for the class of service times used. (Note: If $\mu_i = k$ for $1 \leq i \leq k$, then as $k \rightarrow \infty$, the service time requirement approaches the deterministic service time $X \equiv 1$. By using a more general Markov process to model X , any service time distribution can be approximated, and still N is reversible.)

5.5. A sequence of symmetric star shaped loss networks

Consider a circuit switching system with $M \geq 2$ links, all sharing an endpoint at a central switching node. Suppose the set of routes used by calls is the set of all (unordered) pairs of distinct links. There are thus $\binom{M}{2}$ routes. Suppose that calls arrive for any given route according to a Poisson process of rate $\nu = 4/(M-1)$. Suppose that the call holding times are independent and exponentially distributed with mean one. Finally, suppose that each link has capacity $C = 5$, so that a call offered to a route is blocked and discarded if either of the two links in the route already has 5 calls in progress. (a) What Markov chain describes the system? Describe the state space and transition rates. Is the chain reversible? (b) What is the conservative upper bound on the blocking probability for a given link, and what is Whitt's lower bound on the call acceptance probability. Consider the cases $M = 2, 3, 4$ and $M \rightarrow \infty$. Is the bound exact for any of these values? (c) Derive the reduced load approximation for the blocking probability of a link. Making an independence assumption, derive an approximation for the probability that a call is accepted. Again, consider the cases $M = 2, 3, 4$ and $M \rightarrow \infty$. Comment on the accuracy of the approximation for different values of M .

Note: Solutions are not provided for the remainder of the problems in this section.

5.6. A simple open Jackson network

Consider the Jackson type network (network of single server queues, independent exponentially distributed service times with queue-dependent means, exogenous arrival processes are Poisson and Markov random routing is used) shown in Figure 5.3. The service rates at the three stations are 1, 2, and 1 respectively, and the exogenous arrival rate is ν . At the three points indicated, a customer randomly chooses a path to continue on, each of two choices having probability 0.5. (a) Determine the range of ν for which the system is positive recurrent. (b) Determine the mean time in the network for a typical customer. (c) Each customer exits at either point A or point B. Find the probability that a typical customer exits at point A. (Does your answer depend on ν ?) (d) Indicate how the network can be embedded into an equivalent closed queueing network.

5.7. A tale of two cities

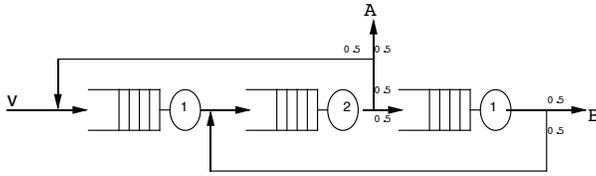


Figure 5.3: A three queue Jackson network

Two cities are connected by two one-way channels, each of capacity C bits per second. Each channel carries packets which are initiated according to a Poisson process at rate a packets per second. Assume that packet lengths are independent, exponentially distributed with mean length μ^{-1} bits. It is proposed that the two channels be replaced by a single half-duplex channel (capable of carrying messages in either direction, but only one direction at a time) of capacity C' . How large must C' be in order to achieve smaller mean waiting times than for the original network? (Model links by single server queues.)

5.8. A tale of three cities

Two communication networks are being considered to link three cities. The first has six one-way channels of capacity C bps, one for each ordered pair of cities. The second has three one-way channels of capacity $2C$ bps each, and the channels connect the three cities in a ring. In the first network packets need only one hop while in the second network some of the packets need travel two hops. Assume that for any ordered pair of cities, packets originate at the first city destined for the second city according to a Poisson process of rate a . Thus there are six types of packets. Assume that packets are exponentially distributed in length with mean length μ^{-1} bits.

- Give a full Markov description for each of these two networks.
- Compute the mean time in the network. (If you need to make an independence approximation at any point, go ahead but state that you've done so.) Under what conditions on a and C will the mean time in the first network be smaller than the mean time in the second network?

5.9. A cyclic network of finite buffer queues

K jobs circulate in a computer pipeline with N stages, as shown in Figure 5.4. Each stage is assumed

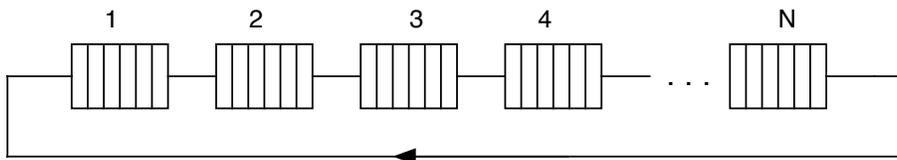


Figure 5.4: A recirculating pipeline

to have one server and storage space for B jobs. Service at any nonempty stage is completed at rate one whenever the next stage is not full, and at rate zero otherwise (this is called "communication blocking"). Define $\eta(N, K, B)$ to be the mean rate at which jobs pass through a given stage in equilibrium. (a) Find $\eta(N, K, B)$ for $B \geq K$. (b) Find $\eta(N, K, 1)$. (Hint: show that all states are equally likely in equilibrium. The rotational symmetry of the system is not enough.) (c) Let $\eta^*(B) = \lim_{N \rightarrow \infty} \max\{\eta(N, K, B) : K \geq 1\}$. Find $\eta^*(+\infty)$ and $\eta^*(1)$. Note: Recent work on interacting particle systems has allowed a determination of $\eta^*(B)$ for all B .

Chapter 6

Calculus of Deterministic Constraints

If a communication link is supposed to guarantee a maximum delay to a traffic flow, then the traffic flow must also satisfy some constraint. In this section a natural calculus of deterministic constraints is examined. In the first subsection the basic (σ, ρ) constraints, and the resulting performance bounds for a single server queue are discussed. In the second section, a more general notion of deterministic constraint is explored. It is shown that token bucket filters are the minimal delay regulators that produce (σ, ρ) -upper constrained traffic streams, and more general regulators are also considered. The third section focuses on applications of service curves, which offer a flexible way to specify a lower bound on the output of a server as a function of the input. Given a constraint on the input traffic and a service curve, a maximum delay can be computed for any server that conforms to the service curve. Two applications are discussed: bounding the maximum delay in a priority queue, and scheduling service at a constant rate link with multiple input streams in order to achieve a specified service curve for each input.

This chapter is based largely on the presentation in C.-S. Chang [5]. The basic calculus of (σ, ρ) constraints was first given in the seminal work of R. L. Cruz [8, 9].

6.1 The (σ, ρ) Constraints and Performance Bounds for a Queue

For simplicity, the case of equal length packets transmitted in discrete time will be considered. A cumulative arrival process A is a nondecreasing, integer-valued function on the nonnegative integers Z_+ such that $A(0) = 0$. For each integer $t \geq 1$, $A(t)$ denotes the total number of arrivals in slots $1, 2, \dots, t$. The number of arrivals at time t , denoted by $a(t)$, is given by $a(t) = A(t) - A(t - 1)$. A cumulative arrival process A is said to be (σ, ρ) -upper constrained if $A(t) - A(s) \leq \sigma + \rho(t - s)$ for any integers $0 \leq s \leq t$. This is denoted as $A \sim (\sigma, \rho)$. In these notes ρ and σ are taken to be integer valued.

A popular way to regulate data streams is through the use of a token bucket filter. A token bucket filter (with no dropping) with parameters (σ, ρ) operates as follows. The filter has a packet queue with unlimited storage capacity, and it has a token bucket (implemented as a counter). The events at an integer time occur in the following order. First new packet arrivals are added into the queue and ρ new tokens are added to the token bucket. Then as many of the packets immediately depart as possible, with the only restriction being that each departing packet take one token. Finally, if there are more than σ tokens in the queue, some tokens are dropped to bring the total number of tokens to σ .

A token bucket filter with parameters (σ, ρ) is a (σ, ρ) regulator, meaning that for any input process A the output process B is (σ, ρ) -upper constrained. Indeed, since at most σ tokens are in the bucket just before time $s + 1$, and since $\rho(t - s)$ additional tokens arrive in slots $s + 1, \dots, t$, at most $\sigma + \rho(t - s)$ packets can depart from the filter in those slots. So the departure process from the token bucket filter is indeed (σ, ρ) -upper constrained. It is shown in the next section that among all FIFO servers producing (σ, ρ) -upper constrained outputs, the token bucket filter requires the least delay for each packet.

If constrained flows are merged, the output process is also constrained:

$$A_i \sim (\sigma_i, \rho_i) \Rightarrow \sum A_i \sim \left(\sum \sigma_i, \sum \rho_i \right)$$

The remainder of this section deals with a queue with a constant rate server, and examines performance bounds implied if the input process is (σ, ρ) -upper constrained. First the queue process is defined. Let C be a positive integer and let A denote a cumulative arrival process. Suppose the arrival process is fed into a queue with a server that can serve up to C packets per slot. Think of the events at an integer time as occurring in the following order. First arrivals, if any, are added to the queue, then up to C customers in the queue are instantaneously served, and then the number of customers in the queue is recorded. Since we take service times to be zero, the amount of time a customer spends in the queue is the same as the amount of time the customer spends in the system, and is simply called the delay of the customer. The queue length process is recursively determined by the Lindley equation:

$$q(t + 1) = (q(t) + a(t + 1) - C)_+$$

with the initial condition $q(0) = 0$. As can be shown by induction on t (and as seen earlier in the course) the solution to the Lindley equation can be written as

$$q(t) = \max_{0 \leq s \leq t} [A(t) - A(s) - C(t - s)] \quad (6.1)$$

for all $t \geq 0$. The cumulative output process B satisfies, by (6.1),

$$\begin{aligned} B(t) &= A(t) - q(t) \\ &= \min_{0 \leq s \leq t} A(s) + C(t - s) \text{ all } t \geq 0. \end{aligned}$$

Suppose now that the input A is (σ, ρ) -upper constrained. Then, if $\rho \leq C$, (6.1) immediately implies that $q(t) \leq \sigma$ for all t . Conversely if $C = \rho$ and $q(t) \leq \sigma$ for all t , then $A \sim (\sigma, \rho)$.

Next, upper bounds on the length of a busy period and on packet delay are derived, assuming that the input A is (σ, ρ) -upper constrained. Given times s and t with $s \leq t$, a busy period is said to begin at s and end at t if $q(s - 1) = 0$, $a(s) > 0$, $q(r) > 0$ for $s \leq r < t$, and $q(t) = 0$. We define the duration B of the busy period to be $B = t - s$ time units. Given such a busy period, there must be C departures at each of the B times $\{s, \dots, t - 1\}$, and at least one packet in the queue at time $t - 1$. Thus at least $CB + 1$ packets must arrive at times in $\{s, \dots, t - 1\}$ in order to "sustain" the busy period. Since A is (σ, ρ) -upper constrained, at most $\sigma + \rho B$ packets can arrive in those slots. Thus it must be that $CB + 1 \leq \sigma + \rho B$ which yields $B \leq \frac{\sigma - 1}{C - \rho}$. But since B is an integer it must be that $B \leq \lfloor \frac{\sigma - 1}{C - \rho} \rfloor$.

The delay of a packet in the queue is the time the packet departs minus the time it arrives. The delay of any packet is less than or equal to the length of the busy period in which the packet arrived. Thus,

$$d \leq \lfloor \frac{\sigma - 1}{C - \rho} \rfloor$$

(for any service order). (Note: If one unit of service time is added, the total system time would be bounded by

$$d + 1 \leq \lfloor \frac{\sigma - 1}{C - \rho} \rfloor + 1 \leq \lceil \frac{\sigma}{C - \rho} \rceil$$

which is the bound given in [5], although [5] uses a different definition of the length of a busy period.

A tighter bound can be given in case the order of service is first in first out (FIFO). If a packet has a nonzero waiting time, then it is carried over from the time it first arrived to the next time slot. The total number of packets carried over, including that packet, is less than or equal to σ , as shown above. Thus, assuming FIFO order, since at least C of these q packets depart in each of the subsequent slots until all q are served, the delay for the packet is less than or equal to d_{FIFO} , given by

$$d_{FIFO} = \lceil \sigma / C \rceil.$$

(Note: This delay also does not include an extra unit of time for service.)

In order to analyze networks, it is important to find constraints on the output of a device, because the output of one device could be the input of another device. A constraint on the output of a device is readily found if the input satisfies a known constraint and if a bound on the maximum delay of the device is known. Suppose that the input A to a device is (σ, ρ) -upper constrained, and that the maximum delay in the device is d . Then for $s < t$, any packets that departs from the device at a time in $\{s + 1, \dots, t\}$ must have arrived at one of the $t - s + d$ times in $\{s + 1 - d, \dots, t\}$. Therefore, $B(t) - B(s) \leq A(t) - A(s - d) \leq \sigma + \rho(t - (s - d)) = \sigma + \rho d + \rho(t - s)$. Therefore, B is $(\sigma + \rho d, \rho)$ -upper constrained.

Consider, for example, a queue with a server of rate $C \geq 1$ and input arrival process A . The cumulative output process B does not depend on the order of service, so no matter what the order of service, the output process is $(\sigma + \rho d_{FIFO}, \rho)$ -upper constrained.

6.2 f -upper constrained processes

The notion of (σ, ρ) constraints has a natural generalization. The generalization is useful not only for the sake of generalization, but it helps in the analysis of even basic (σ, ρ) constrained traffic. Let f be a nondecreasing function from Z_+ to Z_+ . A cumulative arrival process A is said to be f -upper constrained if $A(t) - A(s) \leq f(t - s)$ for all s, t with $0 \leq s \leq t$. For example, if $f(t) = \sigma + \rho t$, then A is f -upper constrained if and only if it is (σ, ρ) -upper constrained.

Rearranging the definition yields that A is f -upper constrained if and only if $A(t) \leq A(s) + f(t - s)$ for $0 \leq s \leq t$. Equivalently, A is f -upper constrained if and only if $A \leq A \star f$, where for two functions f and g defined on Z_+ , $f \star g$ is the function on Z_+ defined by

$$(f \star g)(t) = \min_{0 \leq s \leq t} g(s) + f(t - s). \quad (6.2)$$

Note that $f \star g$ is a convolution in the (\min, plus) algebra. That is, (6.2) uses a minimum rather than addition (as in an integral), and (6.2) uses addition rather than multiplication.

It turns out that some functions f can be reduced, without changing the condition that an arrival process is f -upper constrained. For example, changing $f(0)$ to zero has no effect on the requirement that A is f -upper constrained, because trivially $A(t) - A(t) \leq 0$ anyway. Moreover, suppose A is f -upper constrained and $s, u \geq 0$. Then $A(s + u) - A(s) \leq f(u)$, but a tighter bound may be implied. Indeed, if $n \geq 1$ and u is represented as $u = u_1 + \cdots + u_n$ where $u_i \geq 1$, integer, then

$$\begin{aligned} A(s + u) - A(s) &= (A(s + u_1) - A(s)) + (A(s + u_1 + u_2) - A(s + u_1)) \\ &\quad + \cdots + (A(s + u_1 + \cdots + u_n) - A(s + u_1 + \cdots + u_{n-1})) \\ &\leq f(u_1) + f(u_2) + \cdots + f(u_n) \end{aligned}$$

So $A(t) - A(s) \leq f^*(t - s)$ where f^* , called the *subadditive closure* of f , is defined by

$$f^*(u) = \begin{cases} 0 & \text{if } u = 0 \\ \min\{f(u_1) + \cdots + f(u_n) : n \geq 1, u_i \geq 1 \text{ for each } i, \text{ and } u_1 + \cdots + u_n = u\} & \text{if } u \geq 1 \end{cases}$$

The function f^* has the following properties.

- a) $f^* \leq f$
- b) A is f -upper constrained if and only if A is f^* -upper constrained
- c) f^* is subadditive; $f^*(s + t) \leq f^*(s) + f^*(t)$ all $s, t \geq 0$.
- d) If g is any other function with $g(0) = 0$ satisfying (a) and (c), then $g \leq f^*$.

Recall that the output of a token bucket filter with parameters (σ, ρ) is (σ, ρ) -upper constrained for any input. More generally, given a nondecreasing function f from Z_+ to Z_+ , a *regulator for f* is a device such that for any input A the corresponding output B is f -upper constrained. We shall only consider regulators that output packets only after they arrive, requiring $B \leq A$.

Roughly speaking, regulators (that don't drop packets) delay some packets in order to smooth out the arrival process. There are many possible regulators for a given arrival sequence, but it is desirable for the delay in a regulator to be small. A regulator is said to be a *maximal regulator for f* if the following is true: For any input A , if B is the output of the regulator for input A and if \tilde{B} is a cumulative arrival process such that $\tilde{B} \leq A$ (flow condition) and \tilde{B} is f -upper constrained, then $\tilde{B} \leq B$. Clearly if a maximal regulator for f exists, it is unique.

Proposition 6.2.1 *A maximal regulator for f is determined by the relation $B = A \star f^*$.*

Proof. Let A, B , and \tilde{B} be as in the definition of maximal regulator. Then

$$\tilde{B} \leq \tilde{B} \star f^* \leq A \star f^* = B \tag{6.3}$$

The first equality in (6.3) holds because \tilde{B} is f -upper constrained. The inequality holds because $\star f^*$ is a monotone operation. The final equality holds by the definition of B . ■

Corollary 6.2.2 *Suppose f_1 and f_2 are nondecreasing functions on Z_+ with $f_1(0) = f_2(0) = 0$. Two maximal regulators in series, consisting of a maximal regulator for f_1 followed by a maximal regulator for f_2 , is a maximal regulator for $f_1 \star f_2$. In particular, the output is the same if the order of the two regulators is reversed.*

Proof. Let A be the input to the first regulator and let B be the output of the second regulator. Then

$$B = (A \star f_1^*) \star f_2^* = A \star (f_1^* \star f_2^*) = A \star (f_1 \star f_2)^*, \quad (6.4)$$

where the last equality in (6.4) depends on the assumption that $f_1(0) = f_2(0) = 0$. The first part of the corollary is proved. The second part follows from the uniqueness of maximal regulators and the fact that $f_1 \star f_2 = f_2 \star f_1$. ■

Proposition 6.2.3 *The token bucket filter with parameters (σ, ρ) is the maximal (σ, ρ) regulator.*

Proof. A key step in the proof is to represent the state of a token bucket filter as a function of the queue length process for a queue with constant server rate ρ packets per unit time. Let A denote a cumulative arrival process, and let q denote the corresponding queue length process when A is fed into a queue with a server of constant capacity ρ packets per time unit. Therefore,

$$q(t) = \max_{0 \leq s \leq t} [A(t) - A(s) - \rho(t - s)].$$

Then the size of the packet queue for a (σ, ρ) token bucket filter is $(q(t) - \sigma)_+$, and the number of tokens in the the filter is $(\sigma - q(t))_+$. See Figure This representation can be readily proved by induction on time.

Therefore, if B is the cumulative departure process from a (σ, ρ) token bucket filter with input A , then B is given by

$$\begin{aligned} B(t) &= A(t) - (q(t) - \sigma)_+ \\ &= A(t) - \left(\max_{0 \leq s \leq t} A(t) - A(s) - [\sigma + \rho(t - s)] \right)_+ \\ &= \min \left\{ A(t), \min_{0 \leq s \leq t} A(s) + \sigma + \rho(t - s) \right\} \end{aligned}$$

That is, $B = f^* \star A$ for $f(t) = \sigma + \rho t$, which concludes the proof. ■

6.3 Service Curves

So far we have considered passing constrained processes into a maximal regulator. Examples of maximal regulators include a token bucket filter, or a queue with fixed service rate. The output for these servers is completely determined by the input. In practice, packet streams can be fed into many other types of servers, which often involve interaction among multiple packet streams. Examples include a station in a token ring network or a link in a communication network with a service discipline such as pure priority or weighted round robin. The exact service offered to a particular stream thus cannot be fixed in advance. It is thus useful to have a flexible way to specify a guarantee that a particular server offers to a particular stream. The definition we use naturally complements the notion of f -upper constraints discussed above.

A *service curve* is a nondecreasing function from Z_+ to Z_+ . Given a service curve f , a server is an f -server if for any input A , the output B satisfies $B \geq A \star f$. That is, $B(t) \geq \min_{0 \leq s \leq t} \{A(s) + f(t - s)\}$.

Some comments about the definition are in order. First, note that the notion of f -server involves an inequality, rather than a complete specification of the output given the input as in the case for regulators. Second, in these notes we only consider servers such that the flow constraint $B(t) \leq A(t)$ is satisfied, and it is assumed that $A(0) = 0$ for all cumulative arrival streams. Taking $s = t$ in the above equation implies that $B(t) \geq A(t) + f(0)$. This can only happen if $f(0) = 0$, so that when we discuss f -servers we will only consider functions f with $f(0) = 0$. Third, the definition requires that the output inequality is true for all input processes (it is “universal” in A). In some applications (not discussed in these notes) a server might be an f server only for a specified class of input processes.

Examples For an example of a service curve, given an integer $d \geq 0$, let O_d denote the function

$$O_d(t) = \begin{cases} 0 & \text{for } t \leq d \\ +\infty & \text{for } t > d \end{cases}$$

Then a FIFO device is an O_d -server if and only if the delay of every packet is less than or equal to d , no matter what the input process. A queue with constant service rate C is an f server for $f(t) = Ct$. A leaky bucket regulator is an f server for $f(t) = (\sigma + \rho t)I_{\{t \geq 1\}}$. More generally, the maximal regulator for a function f is an f^* -server.

The following proposition describes some basic performance guarantees for a constrained input process passing through an f -server. Suppose an f_1 -upper constrained process A passes through an f_2 -server. Let $d_V = \max_{t \geq 0} (f_1^*(t) - f_2(t))$ and $d_H = \min\{d \geq 0 : f_1(t) \leq f_2(t + d) \text{ for all } t \geq 0\}$. That is, d_V is the maximum vertical distance that the graph of f_1^* is above f_2 , and d_H is the maximum horizontal distance that the graph of f_2 is to the right of f_1^* .

Proposition 6.3.1 *The queue size $A(t) - B(t)$ is less than or equal to d_V for any $t \geq 0$, and if the order of service is FIFO, the delay of any packet is less than or equal to d_H .*

Proof. Let $t \geq 0$. Since the server is an f_2 -server, there exists an s^* with $0 \leq s^* \leq t$ such that $B(t) \geq A(t - s^*) + f_2(s^*)$. Since A is f_1 -upper constrained, $A(t) \leq A(t - s^*) + f_1^*(s^*)$. Thus, $A(t) - B(t) \leq f_1^*(s^*) - f_2(s^*) \leq d_V$.

To prove the delay bound, suppose a packet arrives at time t and departs at time $\bar{t} > t$. Then $A(t) > B(\bar{t} - 1)$. Since the server is an f_2 -server, there exists an s^* with $0 \leq s^* \leq \bar{t} - 1$ such that $B(\bar{t} - 1) \geq A(s^*) + f_2(\bar{t} - 1 - s^*)$. Because $A(t) > B(\bar{t} - 1)$, it must be that $0 \leq s^* \leq t$. Since A_1 is f_1 -upper constrained, we thus have

$$A(s^*) + f_1(t - s^*) \geq A(t) \geq B(\bar{t} - 1) \geq A(s^*) + f_2(\bar{t} - 1 - s^*),$$

so that $f_1(t - s^*) > f_2(\bar{t} - 1 - s^*)$. Hence, $\bar{t} - 1 - t < d_H$, so that $\bar{t} - t \leq d_H$. ■

Consider a queue with a server of constant rate C which serves input streams 1 and 2, giving priority to packets from input 1, and serving the packets within a single input stream in FIFO order. Let A_i denote the cumulative arrival stream of input i .

Proposition 6.3.2 (See [5, Theorem 2.3.1]). *If A_1 is f_1 -upper constrained, the link is an \tilde{f}_2 -server for the type 2 stream, where $\tilde{f}_2(t) = (Ct - f_1(t))^+$.*

Example For example, suppose A_i is (σ_i, ρ_i) constrained for each i . Then $\tilde{f}_2(t) = ((C - \rho_1)t - \sigma_1)_+$. Applying Proposition 6.3.1 to the input 2 yields that the delay for any packet in input 2 is

less than or equal to $\frac{\sigma_1 + \sigma_2}{C - \rho_1}$. Since packets in input 1 are not affected by the packets from input 2, the delay for any packet in input 1 is less than or equal to $\lceil \frac{\sigma_1}{C} \rceil$.

In comparison, if the queue were served in FIFO order, then the maximum delay for packets from either input is $\lceil \frac{\sigma_1 + \sigma_2}{C} \rceil$. Conclude that if σ_1 is much smaller than σ_2 , the delay for input 1 packets is much smaller for the priority server than for the FIFO server.

The final topic of the section is the problem of scheduling service for multiple input streams on a constant rate link to meet specified service curves for each input. The scheduling algorithm used is the Service Curve Earliest Deadline (SCED) algorithm. Suppose the i th input has cumulative arrival process A_i which is known to be g_i -upper constrained, and suppose that the i th input wants to receive service conforming to a specified service curve f_i . Suppose the link can serve C packets per unit time. Given C , the functions g_i and service curves f_i for all i , the basic question addressed is this. Is there a scheduling algorithm (an algorithm that determines which packets to serve at each time) so that the service for each input A_i conforms to service curve f_i ?

The SCED algorithm works as follows. For each input stream i let $N_i = A_i \star f_i$. Then $N_i(t)$ is the minimum number of type i packets that must depart by time t in order that the input i see service curve f_i . Based on this, a deadline can be computed for each packet of input i . Specifically, the deadline for the k th packet from input i is the minimum t such that $N_i(t) \geq k$. Simply put, if all packets are scheduled by their deadlines, then all service curve constraints are met.

In general, given any arrival sequence with deadlines, if it is possible for an algorithm to meet all the deadlines, then the earliest deadline first (EDF) scheduling algorithm can do it. The SCED algorithm, then, is to put deadlines on the packets as described, and then use earliest deadline first scheduling.

Proposition 6.3.3 *Given g_i, f_i for each i and capacity C satisfying*

$$\sum_{i=1}^n (g_i \star f_i)(t) \leq Ct \text{ all } t. \quad (6.5)$$

the service to each input stream i provided by the SCED scheduling algorithm conforms to service curve f_i .

Proof. Fix a time $t_o \geq 1$. Imagine that all packets with deadline t_o or earlier are colored red, and all packets with deadline $t_o + 1$ or later are white. For any time $t \geq 0$, let $q_o(t)$ denote the number of red packets that are carried over from time t to time $t + 1$. Since EDF is used, red packets have pure priority over white packets, so q_o is the queue length process in case all white packets are simply ignored. It must be proved that $q_o(t_o) = 0$. For $0 \leq s \leq t_o - 1$, the number of red packets that arrive from stream i in the set of times $\{s + 1, \dots, t_o\}$ is $(N_i(t_o) - A_i(s))_+$. Also, $N_i(t_o) \leq A_i(t_o)$. Therefore

$$q_o(t_o) = \max_{0 \leq s \leq t_o} \left\{ \sum_i (N_i(t_o) - A_i(s))_+ \right\} - C(t_o - s).$$

For any s with $1 \leq s \leq t_o$,

$$N_i(t_o) = (A_i \star f_i)(t_o) \leq (A_i \star g_i \star f_i)(t_o) = \min_{0 \leq u \leq t_o} A_i(u) + (g_i \star f_i)(t_o - u) \leq A_i(s) + (g_i \star f_i)(t_o - s)$$

It follows that

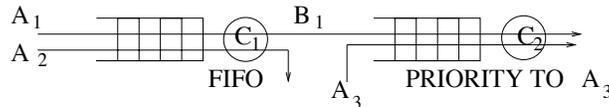
$$\sum_i (N_i(t_o) - A_i(s))_+ \leq \sum_i (g_i \star f_i)(t_o - s) \leq C(t_o - s),$$

so that $q_o(t_o) = 0$, and the proposition is proved. ■

6.4 Problems

6.1. Deterministic delay constraints for two servers in series

Consider the network shown. There are three arrival streams. Suppose that A_i is (σ_i, ρ_i) -upper



constrained for each i and that $\rho_1 + \rho_2 \leq C_1$ and $\rho_1 + \rho_3 \leq C_2$. The first server is a FIFO server, and the second server gives priority to stream 3, but is FIFO within each class. For all three parts below, express your answers in terms of the given parameters, and also give numerical answers for the case $(\sigma_i, \rho_i) = (4, 2)$ for $1 \leq i \leq 3$ and $C_1 = C_2 = 5$.

(a) Give the maximum delay d_1 for customers of the first queue. (Your answer should be finite even if $\rho_1 + \rho_2 = C_1$ due to FIFO service order). (b) Find a value of σ such that B_1 is (σ, ρ_1) -upper constrained. (Seek the smallest value you can.)

(c) Find an upper bound on the delay that stream 1 suffers in the second queue.

6.2. Calculation of an acceptance region based on the SCED algorithm

Let

$$\begin{aligned} g_1(t) &= 20 + t & f_1(t) &= 5(t - 4)_+ \\ g_2(t) &= 8 + 4t & f_2(t) &= 6(t - 20)_+ \end{aligned}$$

(a) Verify that

$$g_i^*(t) = \begin{cases} 0 & \text{if } t = 0 \\ g_i(t) & \text{if } t \geq 1 \end{cases}$$

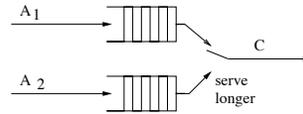
(b) Sketch $g_i^* \star f_i$ for $i = 1, 2$ on two separate graphs. These functions are piecewise linear. Label the breakpoints (places the slopes change), the values of the functions $g_i^* \star f_i$ at the breakpoints, and the values of slope between breakpoints.

(c) Suppose that $n_1 + n_2$ streams share a constant bit rate link of capacity $C = 100$ such that for $i = 1, 2$, there are n_i streams that are g_i -upper constrained, and that each of these n_i streams require the link to appear as an f_i -server. Using Service Curve Earliest Deadline first (SCED), it is possible to accommodate the streams if (n_1, n_2) satisfies $\sum_i n_i g_i^* \star f_i \leq Ct$ for all t . Find and sketch the region of admissible (n_1, n_2) pairs. (Hint: It is enough to check the inequality at the breakpoints and at $t \rightarrow \infty$. Why?)

6.3. Serve longer service priority with deterministically constrained arrival processes

Suppose two queues are served by a constant rate server with service rate C , which serves the longer queue. Specifically, suppose a discrete-time model is used, and after new customers arrive

in a given slot, there are C potential services available, which are allocated one at a time to the longer queue. For example, if after the arrivals in a slot there are 9 customers in the first queue, and 7 in the second, and if $C = 4$, then 3 customers are served from the first queue and 1 from the second. Suppose the arrival process A_i to queue i is (σ_i, ρ_i) -upper constrained, for each i , where $\sigma_1, \sigma_2, \rho_1, \rho_2$, and C are strictly positive integers such that $\rho_1 + \rho_2 \leq C$.



- Find a bound on the maximum number of customers in the system, carried over from one slot to the next, which is valid for any order of service within each substream.
- Find a bound on the maximum delay which is valid for any order of service within each substream. (The bound should be finite if $\rho_1 + \rho_2 < C$.)
- Suppose in addition that customers within each stream are served in FIFO order. Find an upper bound for the delay of customers in stream 1 which is finite, even if $\rho_1 + \rho_2 = C$. Explain your reasoning.

Chapter 7

Graph Algorithms

In class we covered:

- (i) basic graph terminology
- (ii) the Prim-Dijkstra and Kruskal algorithms for the minimum weight spanning tree problem in a directed graph
- (iii) the Bellman-Ford and Dijkstra algorithms for the shortest path problem

Those topics are not covered in these notes: the reader is referred to Bertsekas and Gallager, *Data Networks*, for those topics.

7.1 Maximum Flow Problem

A single-commodity flow network is a collection $N = (s, t, V, E, C)$, where (V, E) is a directed graph, s and t are distinct nodes in V , and $C = (C(e) : e \in E)$ is a vector of nonnegative link capacities. We write $e = (i, j)$ if e is the link from node i to node j , in which case $C(e)$ can also be written as $C(i, j)$. Let $f = (f(e) : e \in E)$ such that $0 \leq f(e) \leq C(e)$ for all links e . The net flow into a node j is given by

$$\sum_{i:(i,j) \in E} f(i, j) - \sum_{k:(j,k) \in E} f(j, k), \quad (7.1)$$

and the net flow out of node j is minus one times the flow into node j . The net flow into a set of nodes is the sum of the net flows into the nodes within the set. If the net flow into each node other than s and t is zero, then f is called an $s - t$ flow. The value of an $s - t$ flow is defined to be the net flow out of node s , or, equivalently, the net flow into node t .

An $s - t$ cut for N is a partition $(A : B)$ of the set of nodes (meaning that $A \cup B = V$ and $A \cap B = \emptyset$) such that $s \in A$ and $t \in B$. The *capacity* of such a cut is $C(A, B) = \sum_{(i,j) \in E: i \in A, j \in B} C(i, j)$. The famous min-flow, max-cut theorem, is stated as follows.

Theorem 7.1.1 (Max-Flow, Min-Cut Theorem) *The maximum value of $s - t$ flows is equal to the minimum capacity of $s - t$ cuts.*

We shall give a proof of the theorem in the remainder of this section. On the one hand, for any $s - t$ flow f and any $s - t$ cut $(A : B)$

$$\begin{aligned} \text{value}(f) &= \sum_{(i,j) \in E: i \in A, j \in B} f(i,j) - \sum_{(j,i) \in E: i \in A, j \in B} f(j,i) \\ &\leq \sum_{(i,j) \in E: i \in A, j \in B} C(i,j) = C(A,B) \end{aligned} \tag{7.2}$$

The inequality in (7.2) is a consequence of the fact that $f(i,j) \leq C(i,j)$ and $f(j,i) \geq 0$ for all $i \in A$ and $j \in B$. Thus, the value of any $s - t$ flow is less than or equal to the capacity of any $s - t$ cut. It remains to be shown that there exists an $s - t$ flow and an $s - t$ cut such that the value of the flow is equal to the capacity of the cut.

Let f be an $s - t$ flow with maximum value. Such a flow exists because the set of $s - t$ flows is compact and $\text{value}(f)$ is a continuous function of f . Without loss of generality, suppose that $E = V \times V$ (simply set $C(e) = 0$ for links not originally in the network.) Call a link (i,j) *good* if $f(i,j) < C(i,j)$ or $f(j,i) > 0$. Intuitively, a link is good if the net link flow $f(i,j) - f(j,i)$ can be increased without violating the capacity constraints. Let A denote the set of nodes i in V such that there is a path from s to i consisting of good links, and let $B = V - A$. Note that $t \notin A$, for otherwise the flow vector f could be modified to yield a flow with strictly larger value. In addition, by the definition of good link and of A , $f(i,j) = C(i,j)$ and $f(j,i) = 0$ for all $i \in A$ and all $j \in B$. Thus, the inequality in (7.2) holds with equality for this specific choice of cut (A,B) . This completes the proof of the min-flow, max-cut theorem.

The above proof suggests an algorithm for computing an $s - t$ flow with maximum value. Starting with an arbitrary flow f , one seeks a path from s to t using only good links. If no such path can be found, the flow has maximum value. If a path is found, a new flow with a strictly larger value can be found. An algorithm can in fact be found (called the Malhotra, Kumar and Maheshwari algorithm) to find a maximum flow in $O(|V|^3)$ steps. A variation of the algorithm can also find a flow with minimum weight, among all flows with a specified value, when nonnegative weights are assigned to the links of the graph, in addition to the capacities.

7.2 Problems

7.1. A shortest path problem

A weighted graph with a single source node s is shown in Fig. 7.1. The weight of each edge is assumed to be the same in either direction. (a) Indicate the tree of shortest paths from s to all other nodes found by Dijkstra's shortest path algorithm. (b) Is the tree of shortest paths from s for the graph unique? *Justify your answer.* (c) How many iterations of the synchronous Bellman-Ford algorithm are required to find shortest length paths from the source node s to all other nodes of this graph?

7.2. A minimum weight spanning tree problem

Consider again the graph of Figure 7.1. Find the minimum weight (undirected) spanning tree found by Dijkstra's MWST algorithm. Is the MWST for this graph unique? *Justify your answer.*

7.3. A maximum flow problem

A flow graph with a source node s and a terminal node t is indicated in Figure 7.2. A line between a pair of nodes indicates a pair of directed links, one in each direction. The capacity of each directed

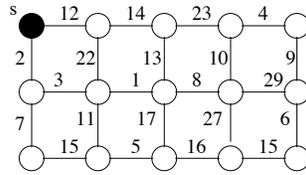


Figure 7.1: An undirected weighted graph.

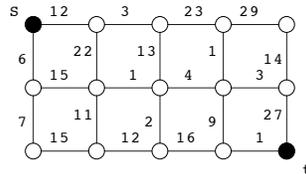


Figure 7.2: A weighted graph with terminals s and t .

link in a pair is the same, and is indicated. Find a maximum value $s - t$ flow and prove that it is indeed a maximum flow.

Chapter 8

Flow Models in Routing and Congestion Control

This chapter gives an introduction to the study of routing and congestion control in networks, using flow models and convex optimization. The role of routing is to decide which routes through the network should be used to get the data from sources to destinations. The role of congestion control is to determine data rates for the users so that fairness among users is achieved, and a reasonable operating point is found on the tradeoff between throughput and delay or loss. The role of flow models is to give a simplified description of network control protocols, which focus on basic issues of stability, rates of convergence, and fairness. Elements of routing and congestion control not covered in this chapter, include routing tables, messages exchanged for maintaining routing tables, packet headers, and so on.

8.1 Convex functions and optimization

Some basic definitions and facts concerning convex optimization are briefly described in this section, for use in the rest of the chapter.

Suppose that Ω is a subset of \mathbb{R}^n for some $n \geq 1$. Also, suppose that Ω is a *convex set*, which by definition means that for any pair of points in Ω , the line segment connecting them is also in Ω . That is, whenever $x, x' \in \Omega$, $ax + (1 - a)x' \in \Omega$ for $0 \leq a \leq 1$. A function f on Ω is a *convex function* if along any line segment in Ω , f is less than or equal to the value of the linear function agreeing with f at the endpoints. That is, by definition, f is a convex function on Ω if $f(ax + (1 - a)x') \leq af(x) + (1 - a)f(x')$ whenever $x, x' \in \Omega$ and $0 \leq a \leq 1$. A *concave function* f is a function f such that $-f$ is convex, and results for minimizing convex functions can be translated to results for maximizing concave functions.

The *gradient*, ∇f , of a function f is a vector valued function, defined to be the vector of first partial derivatives of f :

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

We assume that ∇f exists and is continuous. If $x \in \Omega$ and if v is a vector such that $x + \epsilon v \in \Omega$ for small enough $\epsilon > 0$, then the *directional derivative* of f at x in the direction v is given by

$$\lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon v) - f(x)}{\epsilon} = (\nabla f(x)) \cdot v = \sum_i \frac{\partial f}{\partial x_i}(x) v_i \quad (8.1)$$

We write $x^* \in \arg \min_{x \in \Omega} f(x)$ if x^* minimizes f over Ω , i.e. if $x^* \in \Omega$ and $f(x^*) \leq f(x)$ for all $x \in \Omega$.

Proposition 8.1.1 *Suppose f is a convex function on a convex set Ω with a continuous gradient function ∇f . Then $x^* \in \arg \min_{x \in \Omega} f(x)$ if and only if $(\nabla f(x^*)) \cdot (x - x^*) \geq 0$ for all $x \in \Omega$.*

Proof. (Necessity part.) Suppose $x^* \in \arg \min_{x \in \Omega} f(x)$ and let $x \in \Omega$. Then $x^* + \epsilon(x - x^*) \in \Omega$ for $0 \leq \epsilon \leq 1$, so by the assumption on x^* , $f(x^*) \leq f(x^* + \epsilon(x - x^*))$, or equivalently, $\frac{f(x^* + \epsilon(x - x^*)) - f(x^*)}{\epsilon} \geq 0$. Letting $\epsilon \rightarrow 0$ and using (8.1) with $v = x - x^*$ yields $(\nabla f(x^*)) \cdot (x - x^*) \geq 0$, as required.

(Sufficiency part.) Suppose $x^* \in \Omega$ and that $(\nabla f(x^*)) \cdot (x - x^*) \geq 0$ for all $x \in \Omega$. Let $x \in \Omega$. Then by the convexity of f , $f(\epsilon x + (1 - \epsilon)x^*) \leq \epsilon f(x) + (1 - \epsilon)f(x^*)$ for $0 \leq \epsilon \leq 1$. Equivalently, $f(x) \geq f(x^*) + \frac{f(x^* + \epsilon(x - x^*)) - f(x^*)}{\epsilon}$. Taking the limit as $\epsilon \rightarrow 0$ and using the hypotheses about x^* yields that $f(x) \geq f(x^*) + (\nabla f(x^*)) \cdot (x - x^*) \geq f(x^*)$. Thus, $x^* \in \arg \min_{x \in \Omega} f(x)$. ■

8.2 The Routing Problem

The problem of routing in the presence of congestion is considered in this section. The flow rates of the users are taken as a given, and the problem is to determine which routes should provide the flow. In a later section, we consider the joint routing and congestion control problem, in which both the flow rates of the users and the routing are determined jointly. The following notation is used for both problems.

J is the set links.

R is the set of routes. Each route r is associated with a subset of the set of links. We write $j \in r$ to denote that link j is in route r . If all routes use different sets of links, we could simply consider routes to be subsets of links. But more generally we allow two different routes to use the same set of links.

A is the link-route incidence matrix, defined by $A_{j,r} = 1$ if $j \in r$ and $A_{j,r} = 0$ otherwise.

S is the set of users. Each user s is associated with a subset of the set of routes, which is the set of routes that serve user s . We write $r \in s$ to denote that route r serves user s . We require that the sets of routes for different users be disjoint, so that each route serves only one user.

H is the user-route incidence matrix, defined by $H_{s,r} = 1$ if $r \in s$ and $H_{s,r} = 0$ otherwise.

y_r is the flow on route r .

f_j is the flow on a link j : $f_j = \sum_{r: j \in r} y_r$, or in vector form, $f = Ay$.

x_s is the total flow available to user s : $x_s = \sum_{r \in S} y_r$, or in vector form, $x = Hy$. The vector $x = (x_s : s \in S)$ is fixed for the routing problem, and variable for the joint congestion control and routing problem considered in Section 8.4.

$D_j(f_j)$ is the cost of carrying flow j on link j . The function D_j is assumed to be a convex, continuously differentiable, and increasing function on \mathbb{R}_+ . The cost associated with flow rate f_j on link j is $D_j(f_j)$.

The routing problem, which we now consider, is to specify the route rates for given values of the user rates. It can be written as:

ROUTE(x, H, A, D):

$$\min \sum_{j \in J} D_j \left(\sum_{r: j \in r} y_r \right)$$

subject to

$$x = Hy$$

over

$$y \geq 0.$$

We say that the vector y is a feasible flow meeting demand x if $y \geq 0$ and $x = Hy$. Problem *ROUTE*(x, H, A, D) thus concerns finding a feasible flow y , meeting the demand x , which minimizes the total cost $F(y)$, defined by

$$F(y) = \sum_j D_j \left(\sum_{r: j \in r} y_r \right).$$

As far as the mathematics is concerned, the links could be resources other than connections between nodes, and the set of links in a route could be an arbitrary subset of J . But to be concrete, we can think of there being an underlying graph with nodes representing switches in a network, links indexed by ordered pairs of nodes, and routes forming paths through the network. We could assume each user has a specific origin node and destination node, and the routes that serve a user each connect the source node to the destination node of the user.

A possible choice for the function D_j would be

$$D_j(f_j) = \frac{f_j}{C_j - f_j} + f_j d_j \tag{8.2}$$

where C_j is the capacity of link j and d_j is the propagation delay on link j . If f_j and C_j are in units of packets per second, if arrivals are modeled as a Poisson process, and if service times are modeled as exponentially distributed (based on random packet lengths with an exponential distribution), then a link can be viewed as an $M/M/1$ queue followed by a fixed delay element, and $\frac{1}{C_j - f_j}$ is the mean system time for the queue. Thus, the mean transit time of a packet over the link j , including the propagation delay, is $\frac{1}{C_j - f_j} + d_j$. Then by Little's law, $D_j(f_j)$ given by (8.2), represents the mean number of packets in transit on link j . Therefore, the total cost, $F(y)$, represents the mean number of packets in transit in the network. By Little's law, the average delay of packets entering the network is thus $F(y) / \sum_s x_s$. For fixed user flow rates ($x_s : s \in S$), minimizing the cost is therefore equivalent to minimizing the delay in the network, averaged over all packets.

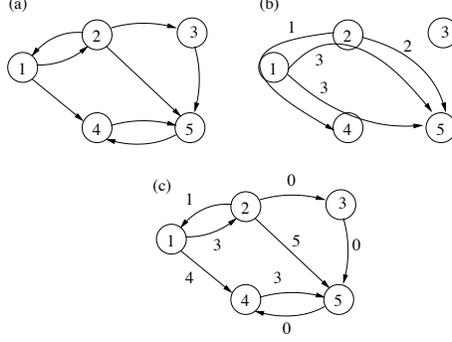


Figure 8.1: Example of a network showing (a) the network, (b) route flows, and (c) link flows.

Example Consider the network graph shown in Figure 8.1(a). For this example there is an underlying set of 5 nodes, numbered 1 through 5. For this example we take

$$J = \{(1, 2), (1, 4), (2, 1), (2, 3), (2, 5), (3, 5), (4, 5), (5, 4)\},$$

with each link labeled by its endpoints.

$$S = \{(1, 5), (2, 4), (2, 5)\},$$

with each user labeled by its origin and destination nodes.

We assume user (1, 5) is served by routes (1,2,5), (1,4,5), (1,2,3,5)

user (2, 4) is served by routes (2,1,4), (2,5,4)

user (2, 5) is served by routes (2,5), (2,3,5)

where each route is labeled by the sequence of nodes that it traverses.

For brevity we write the flow y_r for a route $r = (1, 2, 5)$ as y_{125} , rather than as $y_{(1,2,5)}$. We also write the rate x_s for a user $s = (1, 5)$ as $x_{1,5}$, rather than as $x_{(1,5)}$, and the flow f_j for a link $j = (1, 2)$ as $f_{1,2}$, rather than as $f_{(1,2)}$. An example of an assignment of flow rates to routes, shown in Figure 8.1(b), is given by

$$y_{125} = 3.0, \quad y_{145} = 3.0, \quad y_{1235} = 0, \quad y_{214} = 1.0, \quad y_{254} = 0, \quad y_{25} = 2.0, \quad y_{235} = 0 \quad (8.3)$$

which implies the flow rates to the users

$$x_{1,5} = 6.0, \quad x_{2,4} = 1.0, \quad x_{2,5} = 2.0 \quad (8.4)$$

and the link rates $f_{1,2} = 3, f_{1,4} = 4, f_{2,1} = 1$, etc., as shown in Figure 7.1(c). The resulting cost is

$$F(y) = D_{1,2}(3) + D_{1,4}(4) + D_{2,1}(1) + D_{2,3}(0) + D_{2,5}(5) + D_{3,5}(0) + D_{4,5}(3) + D_{5,4}(0).$$

We consider, in the context of this example, the effect on the cost of changing the flow rates. If y_{125} were increased to $y_{125} + \epsilon$ for some $\epsilon > 0$, then the cost $F(y)$ would increase, with the amount of the increase given by

$$\begin{aligned} \Delta F &= D_{1,2}(3 + \epsilon) - D_{1,2}(3) + D_{2,5}(5 + \epsilon) - D_{2,5}(5) \\ &= \epsilon(D'_{1,2}(3) + D'_{2,5}(5)) + o(\epsilon) \end{aligned}$$

Similarly, if y_{145} were increased by ϵ , the cost would increase by $\epsilon(D'_{1,4}(4) + D'_{4,5}(3))\epsilon + o(\epsilon)$. If instead, y_{145} were decreased by ϵ , the cost would decrease by $\epsilon(D'_{1,4}(4) + D'_{4,5}(3))\epsilon + o(\epsilon)$. If two

small changes were made, the effect on $F(y)$, to first order, would be the sum of the effects. Thus, if y_{125} were increased by ϵ and y_{145} were simultaneously decreased by ϵ , that is, if ϵ units of flow were *deviated* to route (1, 2, 5) from route (1, 4, 5), then the change in cost would be

$$(\text{cost of } 145 \rightarrow 125 \text{ flow deviation}) = \epsilon(D'_{1,2}(3) + D'_{2,5}(5) - D'_{1,4}(4) - D'_{4,5}(3)) + o(\epsilon)$$

That is, to first order, the change in cost due to the flow deviation is ϵ times the first derivative length $D'_{1,2}(3) + D'_{2,5}(5)$ of route (1, 2, 5) minus the first derivative length $D'_{1,4}(4) + D'_{4,5}(3)$ of route (1, 4, 5). Thus, for small enough ϵ , the flow deviation will decrease the cost if flow is deviated from a longer route to a shorter one, if the lengths of the routes are given by the sums of the first derivative link lengths, $D_j(f_j)$, of the links in the routes.

We return from the example to the general routing problem. The observation made about flow deviation in the example is general. It implies that a necessary condition for y to be optimal is that for any s and any $r \in s$, the rate y_r is strictly positive only if r has minimum first derivative length among the routes serving s . Since the set of feasible rate vectors y meeting the demand x is a convex set, and the cost function F is convex, it follows from elementary facts about convex optimization, described in Section 8.1, that the first order necessary condition for optimality is also sufficient for optimality. Specifically, the following is implied by Proposition 8.1.1.

Proposition 8.2.1 *A feasible flow $y = (y_r : r \in R)$ meeting demand x minimizes the cost over all such flows, if and only if there exists a vector $(\lambda_s : s \in S)$ so that $\sum_{j \in r} D'_j(f_j) \geq \lambda_s$, with equality if $y_r > 0$, whenever $r \in s \in S$.*

Given a flow vector y , let δ_r denote the first derivative length of route r . That is, $\delta_r = \sum_{j \in r} D_j(\sum_{r': j \in r'} y_{r'})$. Note that if the rate requirement for a user s is strictly positive, i.e. $x_s > 0$, then the parameter λ_s in Proposition 8.2.1 must satisfy $\lambda_s = \min_{r \in s} \delta_r$. The example and Proposition 8.2.1 suggest the following algorithm, called the flow-deviation algorithm, for finding an optimal flow y . The algorithm is iterative, and we describe one iteration of it.

The flow deviation algorithm One iteration is described. At the beginning of an iteration, a feasible flow vector y , meeting the specified demand x , is given. The iteration finds the next such vector, \bar{y} .

Step 1. Compute the link flow f_j and first derivative link lengths, $D_j(f_j)$, for $j \in J$.

Step 2. For each user s , find a route r_s^* serving s with the minimum first derivative link length.

Step 3. Let $\alpha \in [0, 1]$. For each user s , deviate a fraction α of the flow from each of the other routes serving s to r_s^* . That is, for each s , let

$$\bar{y}_r = \begin{cases} (1 - \alpha)y_r & \text{if } r \in s \text{ and } r \neq r_s^* \\ y_r + \alpha(\sum_{r': r' \neq r \in s} y_{r'}) & \text{if } r = r_s^* \end{cases}$$

Adjust α to minimize $F(\bar{y})$. The resulting vector \bar{y} is the result of the iteration.

The flow deviation algorithm is not a distributed algorithm, but some parts of it can be implemented in a distributed way. A user could gather the first derivative link lengths of its routes

by having the links along each route signal the information in passing packets. A user could then determine which of its routes has the shortest first derivative link length. Synchronization of the flow deviation step across users, and in particular, the fact that all users must collectively determine a single best value of α , is the main obstacle to distributed implementation of the flow deviation algorithm presented. However, by having each user respond slowly enough, possibly asynchronously, convergence can be achieved.

It may give insight into the problem to ignore the step size problem, find an algorithm, and then address the step size selection problem later. This suggests describing an algorithm by an ordinary differential equation. For the routing problem at hand, imagine that all users gradually shift their flow towards routes with smaller first derivative link length. Let $y(t)$ denote the vector of flows at time t , and let $\delta_r(t)$ denote the corresponding first derivative link length of route r at time t , for each $r \in R$. Assume that the users adjust their rates slowly enough that the variables $\delta_r(t)$ can be continuously updated. In the following, we suppress the variable t in the notation, but keep in mind that y and $(\delta_r : r \in R)$ are time varying. We've seen that if r and r' are two routes for a given user s , if $\delta_{r'} > \delta_r$, and if $y_{r'} > 0$, then the cost $F(y)$ can be reduced by deviating flow from r' to r . If the speed of deviation is equal to $y_{r'}(\delta_{r'} - \delta_r)$, then $y_{r'}$ is kept nonnegative, and the factor $(\delta_{r'} - \delta_r)$ avoids discontinuities in case two routes have nearly the same length. Suppose such changes are made continuously for all pairs of routes. Let $\theta = (\theta_{r,r'} : r, r' \in R)$ be a matrix of constants such that $\theta_{r,r'} = \theta_{r',r} > 0$ whenever $r, r' \in s$ for some $s \in S$, and $\theta_{r,r'} = 0$ if r and r' serve different users. We will use the constant $\theta_{r,r'}$ as the relative speed for deviation of flow between routes r and r' . The following equation gives a flow deviation algorithm in ordinary differential equation form.

A continuous flow deviation algorithm

$$\dot{y}_r = \sum_{r' \in s} \{ \theta_{r',r} y_{r'} (\delta_{r'} - \delta_r)_+ - \theta_{r,r'} y_r (\delta_r - \delta_{r'})_+ \} \quad \text{for } r \in s \in S \quad (8.5)$$

Because the changes in y are due to transfers between routes used by the same user, we expect $\dot{x}(t) = Hy(t)$, to be constant. Indeed, using the symmetry of θ and interchanging the variables r and r' on a summation, yields that for any user s ,

$$\begin{aligned} \dot{x}_s = \sum_{r \in s} \dot{y}_r &= \sum_{r \in s} \sum_{r' \in s} \theta_{r',r} y_{r'} (\delta_{r'} - \delta_r)_+ - \sum_{r \in s} \sum_{r' \in s} \theta_{r,r'} y_r (\delta_r - \delta_{r'})_+ \\ &= \sum_{r \in s} \sum_{r' \in s} \theta_{r',r} y_{r'} (\delta_{r'} - \delta_r)_+ - \sum_{r \in s} \sum_{r' \in s} \theta_{r',r} y_{r'} (\delta_{r'} - \delta_r)_+ \\ &= 0 \end{aligned}$$

Now $\frac{\partial F}{\partial y_r} = \delta_r$, so by the chain rule of differentiation, and the same interchange of r and r' used in

computing \dot{x}_s ,

$$\begin{aligned}
\frac{dF(y(t))}{dt} &= \sum_r \delta_r \dot{y}_r \\
&= \sum_{r \in R} \sum_{r' \in R} \{ \theta_{r',r} y_{r'} \delta_r (\delta_{r'} - \delta_r)_+ - \theta_{r,r'} y_r \delta_r (\delta_r - \delta_{r'})_+ \} \\
&= \sum_{r \in R} \sum_{r' \in R} \{ \theta_{r',r} y_{r'} \delta_r (\delta_{r'} - \delta_r)_+ - \theta_{r,r'} y_r \delta_{r'} (\delta_{r'} - \delta_r)_+ \} \\
&= - \sum_{r \in R} \sum_{r' \in R} \theta_{r',r} y_{r'} (\delta_{r'} - \delta_r)_+^2 \\
&\leq 0
\end{aligned}$$

Thus, the cost $F(y(t))$ is decreasing in time. Moreover, the rate of decrease is strictly bounded away from zero away from a neighborhood of the set of y satisfying the necessary conditions for optimality. It follows that all limit points of the trajectory $(y(t) : t \geq 0)$ are solutions to the routing problem, and $F(y(t))$ converges monotonically to the minimum cost.

The step size problem is an important one, and is not just a minor detail. If the users do not implement some type of control on how fast they make changes, oscillations could occur. For example, one link could be lightly loaded one minute, so the next, many users could transfer flow to the link, to the point it becomes overloaded. Then they may all transfer flow away from the link, and the link may be very lightly loaded again, and so on. Such oscillations have limited the use of automated dynamic routing based on congestion information. In applications, an important facet of the implementation is that the flow rates at the links often have to be estimated, and the accuracy of an estimate over an interval of length t is typically on the order of $1/\sqrt{t}$, by the central limit theorem.

The two routing algorithms considered in this section are considered to be *primal* algorithms, because the flow rates to be found are directly updated. In contrast, in a dual algorithm, the primal variables are expressed in terms of certain dual variables, or prices, and the algorithm seeks to identify the optimal values of the dual variables. Dual algorithms arise naturally for networks with hard link constraints of the form $f_j \leq C_j$, rather than with the use of soft link constraints based on cost functions $D_j(f_j)$ as in this section. See the remark in Section 8.7

8.3 Utility Functions

Suppose a consumer would like to purchase or consume some commodity, such that an amount x of the commodity has utility or value $U(x)$ to the consumer. The function U is called the utility function. Utility functions are typically nondecreasing, continuously differentiable, concave functions on \mathbb{R}_+ . Commonly used examples include:

- (i) $U(x) = \beta \log x$ (for some constant $\beta > 0$)
- (ii) $U(x) = \left(\frac{\beta}{1-\alpha} \right) (x^{1-\alpha} - 1)$ (for some constants $\alpha, \beta > 0, \alpha \neq 1$)
- (iii) $U(x) = \frac{\beta}{2} [\bar{x}^2 - (\bar{x} - x)_+^2]$ (for some constants $\beta, \bar{x} > 0$)
- (iv) $U(x) = -\beta \exp(-\gamma x)$ (for some constant $\gamma > 0$)

The function in (ii) converges to the function in (i) as $\alpha \rightarrow 1$.

If x is increased by a small amount ϵ the utility $U(x)$ is increased by $\epsilon U'(x) + o(\epsilon)$. Thus $U'(x)$ is called the *marginal utility* per unit of commodity. Concavity of U is equivalent to the natural property that the marginal utility is monotone nonincreasing in x .

If a rational consumer can buy commodity at a price λ per unit, then the consumer would choose to purchase an amount x to maximize $U(x) - x\lambda$. Differentiation with respect to x yields that the optimal value x_{opt} satisfies

$$U'(x_{opt}) \leq \lambda, \quad \text{with equality if } x_{opt} > 0. \quad (8.6)$$

It is useful to imagine increasing x , starting from zero, until the consumer's marginal utility is equal to the price λ . If U is strictly concave, and if $(U')^{-1}$ is the inverse function of U' , we see that the response of the consumer to price λ is to purchase $x_{opt} = (U')^{-1}(\lambda)$ units of commodity.

The response function for a logarithmic utility function $U(x) = \beta \log x$ has a particularly simple form, and an important interpretation. The response of a rational consumer with such a utility function to a price λ is to purchase x_{opt} given by $x_{opt} = \beta/\lambda$. So $\lambda x_{opt} = \beta$ for any $\lambda > 0$. That is, such a consumer effectively decides to make a payment of β , no matter what the price per unit flow λ is.

Exercise: Derive the response functions for the other utility functions given in (i)-(iv) above.

8.4 Joint Congestion Control and Routing

In Section 8.2, the flow requirement x_s for each user s is assumed to be given and fixed. However, if the values of the x_s 's are too large, then the congestion in the network, measured by the sum of the link costs, may be unreasonably large. Perhaps no finite cost solution exists. Congestion control consists of restricting the input rates to reasonable levels. One approach to doing this fairly is to allow the x_s 's to be variables; and to assume utility functions U_s are given. The joint congestion control and routing problem is to select the y_r 's, and hence also the x_s 's, to maximize the sum of the utilities of the flows minus the sum of link costs. We label the problem the system problem, because it involves elements of the system ranging from the users' utility functions to the topology and cost functions of the network. It's given as follows:

SYSTEM(U, H, A, D) (for joint congestion control and routing with soft link constraints):

$$\max \sum_{s \in S} U_s(x_s) - \sum_{j \in J} D_j \left(\sum_{r: j \in r} y_r \right)$$

subject to

$$x = Hy$$

over

$$x, y \geq 0.$$

This problem involves maximizing a concave objective function over a convex set. Optimality conditions, which are both necessary and sufficient for optimality, can be derived for this joint congestion control and routing problem in a fashion similar to that used for the routing problem

alone.¹ The starting point is to note that the change in the objective function due in increasing x_r by ϵ , for some user s and some route r serving s , is

$$(U'_s(x_s) - \sum_{j \in r} D'_j(f_j))\epsilon + o(\epsilon)$$

This implies that if y is an optimal flow, then

$$U'_s(x_s) \leq \sum_{j \in r} D'_j(f_j), \quad \text{with equality if } y_r > 0, \quad \text{whenever } r \in s \in S. \quad (8.7)$$

That is, condition (8.7) is necessary for optimality of x . Due to the concavity of the objective function, the optimality condition (8.7) is also sufficient for optimality, and indeed Proposition 8.1.1 implies that (8.7) is both necessary and sufficient for optimality.

The first derivative length of a link j , $D'_j(f_j)$, can be thought of as the marginal price (which we will simply call price, for brevity) for using link j , and the sum of these prices over $j \in r$ for a route r , is the price for flow along the route. In words, (8.7) says the following for the routes r serving a user s : The price of any route with positive flow is equal to the marginal utility for s , and the price of any route with zero flow is greater than or equal to the marginal utility for s .

8.5 Hard Constraints and Prices

In some contexts the flow f on a link is simply constrained by a capacity: $f \leq C$. This can be captured by a cost function which is zero if $f \leq C$ and infinite if $f > C$. However, such a cost function is highly discontinuous. Instead, we could consider for some small $\epsilon, \delta > 0$ the function

$$D(f) = \frac{1}{2\epsilon}(f - (C - \delta))_+^2.$$

The price $D'(f)$ is given by

$$D'(f) = \frac{1}{\epsilon}(f - (C - \delta))_+. \quad (8.8)$$

As $\epsilon, \delta \rightarrow 0$, the graph of the price function D' converges to a horizontal segment up to C and a vertical segment at C , as shown in Figure 8.2. That is, for the hard constraint $f \leq C$, the price is zero if $f < C$ and the price is an arbitrary nonnegative number if $f = C$. (This is the complementary slackness condition for Kuhn-Tucker constraints.)

The routing problem formulated in Section 8.2 already involves hard constraints—not inequality constraints on the links—but equality constraints on the user rates x_s . The variable λ_s can be viewed as a marginal price for the rate allocation to user s . The rate x_s of user s could be increased by ϵ at cost $\lambda_s\epsilon + o(\epsilon)$ by increasing the flow along one of the routes r serving s with minimum first derivative link length λ_s .

The joint routing and congestion control problem with hard link constraints is similar to problem *SYSTEM*(U, H, A, D), but with the cost functions ($D_j : j \in J$) replaced by a vector of capacities,

¹In fact, through the introduction of an additional imaginary link, corresponding to undelivered flow, it is possible to reduce the joint congestion control and routing problem described here to a pure routing problem. See [2, Section 6.5.1].

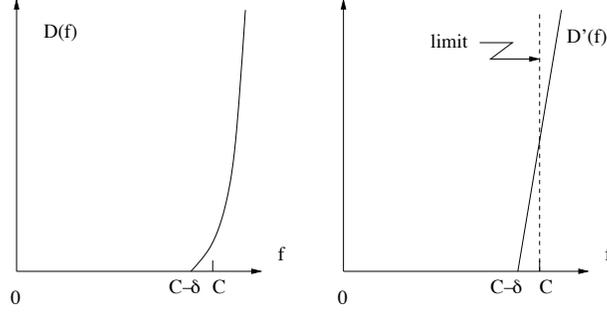


Figure 8.2: Step cost function, approximating a hard constraint.

$C = (C_j : j \in J)$:

SYSTEM(U, H, A, C) (for joint congestion control and routing with hard link constraints):

$$\max \sum_{s \in S} U_s(x_s)$$

subject to

$$x = Hy, \quad Ay \leq C$$

over

$$x, y \geq 0.$$

The optimality conditions for *SYSTEM*(U, H, A, C) are, for some vector $\mu = (\mu_j : j \in J)$,

$$\mu_j \geq 0, \quad \text{with equality if } f_j < C_j, \quad \text{for } j \in J \quad (8.9)$$

$$U'_s(x_s) \leq \sum_{j \in r} \mu_j, \quad \text{with equality if } y_r > 0, \quad \text{whenever } r \in s \in S. \quad (8.10)$$

Equation (8.10) is identical to (8.7), except μ_j replaces $D'_j(f_j)$. Equation (8.9) shows that μ_j satisfies the limiting behavior of the approximating function $D'(f_j)$ in (8.8), as $\epsilon, \delta \rightarrow 0$. It is plausible that (8.9) and (8.10) give optimality conditions for *SYSTEM*(U, H, A, C), because in a formal sense they are the limit of (8.7) as the soft constraints approach hard constraints. The sufficiency of these conditions for optimality is proved in an exercise, based on use of a Lagrangian.

8.6 Decomposition into Network and User Problems

A difficulty of the joint congestion control and routing problem is that it involves the users' utility functions, the topology of the network, and the link cost functions or capacities. Such large, heterogeneous optimization problems can often be decoupled, using prices (a.k.a. Lagrange multipliers) across interfaces. A particularly elegant formulation of this method was given in the work of Kelly [24], followed here. See Figure 8.3. The idea is to view the problem faced by a user as how much to pay. Once each user s has decided to pay β_s for its rate, the network mechanisms can ignore the real utility function $U_s(x_s)$ and pretend the utility function of user s is the surrogate utility function

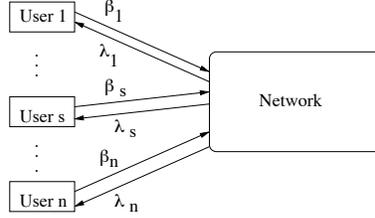


Figure 8.3: Coupling between the network and user problems in the system decomposition.

$V_s(x_s) = \beta_s \log(x_s)$. That is because, as noted in Section 8.3, the response of a user with utility function $\beta_s \log(x_s)$ to a fixed price λ_s is to pay β_s , independently of the price. The network can determine (x, y) , using the surrogate valuation functions. In addition, for each user s , the network can determine a price λ_s , equal to the minimum marginal cost of the routes serving s :

$$\lambda_s = \min_{r \in s} \delta_r \quad \text{where } \delta_r = \begin{cases} \sum_{j \in r} D_j(f_j) & \text{in the case of soft link constraints} \\ \sum_{j \in r} \mu_j & \text{in the case of hard link constraints.} \end{cases} \quad (8.11)$$

The network reports λ_s to each user s . User s can then also learn the allocation rate x_s allocated to it by the network, because $x_s = \lambda_s \beta_s$. The other side of the story is the user optimization problem. A user s takes the price λ_s as fixed, and then considers the possibility of submitting a new value of the payment, β_s , in response. Specifically, if the price λ_s resulting from payment β_s is less than the marginal utility $U'_s(\beta_s/\lambda_s)$, then the user should spend more (increase β_s). Note that the network and user optimization problems are connected by the payments declared by the users (the β_s 's) and the prices declared by the network (the λ_s 's). The network need not know the utility functions of the users, and the users need not know the topology or congestion cost functions of the network.

Input data for the network problem includes the vector of payments β . but not the utility functions. The problem is as follows.

NETWORK(β, H, A, D) (for joint congestion control and routing with soft link constraints):

$$\max \sum_{s \in S} \beta_s \log(x_s) - \sum_{j \in J} D_j \left(\sum_{r: j \in r} y_r \right)$$

subject to

$$x = Hy$$

over

$$x, y \geq 0.$$

The user problem for a given user s does not involve the network topology. It is as follows.

$USER_s(U_s, \lambda_s)$:

$$\max U_r \left(\frac{\beta_s}{\lambda_s} \right) - \beta_s$$

over

$$\beta_s \geq 0.$$

Let us prove that if the optimality conditions of both the user problem and the network problem are satisfied, then the optimality conditions of the system problem are satisfied. We first consider the problem with soft link constraints. The optimality conditions for the user problem are:

$$U'_s(x_s) \leq \lambda_s, \quad \text{with equality if } x_s > 0, \quad \text{for } s \in S. \quad (8.12)$$

The optimality conditions for the network problem are given by (8.7) with U'_s replaced by V'_s . Since $V'_s(x_s) = \frac{\beta_s}{x_s}$ (for this to always be true, we interpret $\frac{\beta_s}{x_s}$ to be zero if $\beta_s = x_s = 0$), the optimality conditions for the network problem are thus:

$$\frac{\beta_s}{x_s} \leq \sum_{j \in r} D'_j(f_j), \quad \text{with equality if } y_r > 0, \quad \text{whenever } r \in s \in S. \quad (8.13)$$

The fact, true by (8.12), that $U'_s(x_s) \leq \lambda_s$ for all $s \in S$, and the rule (8.11) for selecting λ_s , imply that $U'_s(x_s) \leq \sum_{j \in r} D'_j(f_j)$ whenever $r \in s \in S$. Thus, it remains to prove the equality condition in (8.7) in case $y_r > 0$. If $r \in s \in S$ and $y_r > 0$, then $x_s > 0$. Therefore, by (8.13),

$$\lambda_s = \frac{\beta_s}{x_s} \quad (8.14)$$

Also, equality holds in both (8.12) and (8.13), so using (8.14) to eliminate λ_s , we find equality holds in (8.7).

The story is similar for the case of hard link constraints, as we see next. The network problem is

$NETWORK(\beta, H, A, C)$ (for joint congestion control and routing with hard link constraints):

$$\max \sum_{s \in S} \beta_s \log(x_s)$$

subject to

$$x = Hy, \quad Ay \leq C$$

over

$$x, y \geq 0.$$

and the same user problem is used. The optimality condition for the user problem is again (8.12). The optimality condition for the network problem is given by (8.9) and (8.10) with U'_s replaced by V'_s :

$$\mu_j \geq 0, \quad \text{with equality if } f_j < C_j, \quad \text{for } j \in J \quad (8.15)$$

$$\frac{\beta_s}{x_s} \leq \sum_{j \in r} \mu_j, \quad \text{with equality if } y_r > 0, \quad \text{whenever } r \in s \in S. \quad (8.16)$$

We wish to show that the user problem optimality conditions, (8.12), the network problem optimality conditions, (8.15) and (8.16), and the choice of $(\lambda_s : s \in S)$, given in (8.11), imply the optimality conditions for the system problem, (8.9) and (8.10). Equation (8.15) is the same as (8.9). The proof of (8.16) follows by the same reasoning used for soft constraints.

In summary, whether the link constraints are soft or hard, if the network computes the prices using $\lambda_s = \min_{r \in S} \delta_r$, and if the optimality conditions of the network and user problems are satisfied, then so are the optimality conditions of the system problem.

8.7 Specialization to pure congestion control

A model for pure congestion control is obtained by assuming the users in the joint congestion control and routing problem are served by only one route each. In this situation we can take the set of routes R to also be the set of users., and write x_r for the rate allocated to user r . If hard link constraints are used, the congestion control problem is the following.

SYSTEM(U, A, D) (for congestion control with soft link constraints):

$$\max \sum_{r \in R} U_r(x_r) - \sum_{j \in J} D_j(\sum_{r: j \in r} x_r)$$

over

$$x \geq 0.$$

The optimality conditions for this problem are

$$U'_r(x_r) \leq \sum_{j \in r} D'_j(f_j), \quad \text{with equality if } x_r > 0, \quad \text{for all } r \in R. \quad (8.17)$$

If hard link constraints are used, the congestion control problem is the following.

SYSTEM(U, A, C) (for congestion control with hard link constraints):

$$\max \sum_{r \in R} U_r(x_r)$$

subject to

$$Ax \leq C$$

over

$$x \geq 0.$$

The optimality conditions for this problem are

$$\mu_j \geq 0, \quad \text{with equality if } f_j < C_j, \quad \text{for } j \in J \quad (8.18)$$

$$U'_r(x_r) \leq \sum_{j \in r} \mu_j, \quad \text{with equality if } x_r > 0, \quad \text{for } r \in R. \quad (8.19)$$

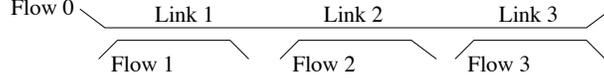


Figure 8.4: A three link network with four users

The corresponding network problem is obtained by replacing the utility function $U_r(x_r)$ by $\beta_r \log(x_r)$ for each r :

NETWORK(β, A, C) (for congestion control with hard link constraints):

$$\max \sum_{r \in R} \beta_r \log(x_r)$$

subject to

$$Ax \leq C$$

over

$$x \geq 0.$$

The optimality conditions for this problem are special cases of (8.18) and (8.19) and are given by (with $f = Ax$):

$$\mu_j \geq 0, \quad \text{with equality if } f_j < C_j, \quad \text{for } j \in J \quad (8.20)$$

$$\beta_r = x_r \cdot \left(\sum_{j \in r} \mu_j \right), \quad \text{for } r \in R, \quad (8.21)$$

The user problem for the congestion control system problem is the same as for the joint routing and congestion control problems. Basically, the user cannot distinguish between these two problems.

Remark We remark briefly about dual algorithms, which seek to find an optimal value for the vector of dual variables, μ . The primal variables, x , are determined by μ through (8.21). The dual variable μ_j is adjusted in an effort to make sure that $f_j \leq C_j$, and to satisfy the other optimality condition, (8.20). Intuitively, if for the current value $\mu(t)$ of the dual variables, the constraint $f_j \leq C_j$ is violated, μ_j should be increased. If the constraint f_j is strictly smaller than C_j , then μ_j should be decreased (if it is strictly positive) or be held at zero otherwise.

Example Consider the following instance of the problem *NETWORK*(β, A, C), for congestion control with hard link constraints. Suppose $J = \{1, 2, 3\}$ and $R = \{0, 1, 2, 3\}$, so there are three links and four users, with one route each. Suppose the route of user 0 contains all three links, and the route of user i contains only link i , for $1 \leq i \leq 3$, as shown in Figure 8.4. Thus, the optimality conditions are given by (8.20) and (8.21). In order for x to be an optimal flow, it is necessary for this example that $f_j = C_j$ for all j , because each link has a single-link user. Therefore, μ_1, μ_2 , and μ_3 can be any values in \mathbb{R}_+ , and $x_i = 1 - x_0$ for $1 \leq i \leq 3$. Condition (8.21) becomes:

$$\beta_0 = x_0(\mu_1 + \mu_2 + \mu_3), \quad \beta_1 = x_1\mu_1, \quad \beta_2 = x_2\mu_2, \quad \beta_3 = x_3\mu_3..$$

Eliminating the μ 's and using $x_i = 1 - x_0$ yields:

$$\beta_0 = x_0 \left(\frac{\beta_1}{1 - x_0} + \frac{\beta_2}{1 - x_0} + \frac{\beta_3}{1 - x_0} \right),$$

or

$$x_0 = \frac{\beta_0}{\beta_0 + \beta_1 + \beta_2 + \beta_3}.$$

For example, if $\beta_i = 1$ for all i , then $x = (0.25, 0.75, 0.75, 0.75)$, and $\mu = (1.333, 1.333, 1.333)$. If β_3 were increased to two, still with $\beta_0 = \beta_1 = \beta_2 = 1$, then the new allocations and link prices would be $x = (0.2, 0.8, 0.8, 0.8)$ and $\mu = (1.25, 1.25, 2.5)$. Thus, the increase in β_3 causes an increase in the allocation to user 3 from 0.75 to 0.8. A side effect is that the rates of users 1 and 2 also increase the same amount, the rate of user 0 is decreased, the price of link 3 increases substantially, while the prices of links 1 and 2 decrease somewhat.

8.8 Fair allocation

The problem $SYSTEM(U, A, C)$ is an example of a pure allocation problem with no costs. The allocation x is constrained to be in the convex set $\{x \geq 0 : Ax \leq C\}$. A generalization is to replace the space of x by a general convex set Λ . For example, we could let $\Lambda = \{x \geq 0 : \sum_{r \in R} x_r^2 \leq C\}$. This leads to the following generalization of $SYSTEM(U, A, C)$:

$ALLOCATE(U, \Lambda)$

$$\max \sum_{r \in R} U_r(x_r)$$

over

$$x \in \Lambda.$$

An optimal solution exists if Λ is closed and bounded and the functions U_r are continuous. By Proposition 8.1.1, x^* is optimal if and only if

$$\sum_{r \in R} U_r'(x_r^*)(x_r - x_r^*) \leq 0 \quad \text{for all } x \in \Lambda. \quad (8.22)$$

In case $U_r(x_r) = \beta_r \log x_r$ for all r , for some constants $\beta_r > 0$, the optimality condition (8.22) becomes

$$\sum_{r \in R} \beta_r \left\{ \frac{x_r - x_r^*}{x_r^*} \right\} \leq 0 \quad \text{for all } x \in \Lambda. \quad (8.23)$$

The term in braces in (8.23) is the weighted normalized improvement of x_r over x_r^* for user r . For example, if $\frac{x_r - x_r^*}{x_r^*} = 0.5$, we say that x_r offers factor 0.5 increase over x_r^* . The optimality condition (8.23) means that no vector x can offer a positive weighted sum of normalized improvement over x^* . A vector x^* that satisfies (8.23) is said to be a *proportionally fair* allocation, with weight vector β .

If instead, $U_i(x) = \left(\frac{\beta_i}{1 - \alpha} \right) (x^{1 - \alpha} - 1)$ for each i , where $\beta_i > 0$ for each i , and $\alpha \geq 0$ is the same for all users, then an optimal allocation is called α -fair, and the optimality condition for α -fairness

is

$$\sum_{i=1}^n \beta_i \left\{ \frac{x_i - x_i^*}{(x_i^*)^\alpha} \right\} \leq 0 \text{ for all } x \in \Lambda.$$

The limiting case $\alpha = 1$ corresponds to proportional fairness. The case $\alpha = 0$ corresponds maximizing a weighted sum of the rates. A limiting form of α fairness as $\alpha \rightarrow \infty$ is max-min fairness, discussed next.

Max-Min Fair Allocation

In the remainder of this section we discuss max min fairness, an alternative to the notion of proportional fairness. Perhaps the easiest way to define the max-min fair allocation is to give a recipe for finding it. Suppose that Λ is a closed, convex subset of \mathbb{R}_+^n containing the origin. Let e_r denote the vector in \mathbb{R}^n with r^{th} coordinate equal to one, and other coordinates zero. Given $x \in \Lambda$, let $g_r(x) = 1$ if $x + \epsilon e_r \in \Lambda$ for sufficiently small $\epsilon > 0$, and $g_r(x) = 0$ otherwise. Equivalently, $g(x)$ is the maximal vector with coordinates in $\{0, 1\}$ such that $x + \epsilon g(x) \in \Lambda$ for sufficiently small $\epsilon > 0$. Construct the sequence $(x^{(k)} : k \geq 0)$ as follows. Let $x^{(0)} = 0$. Given $x^{(k)}$, let $x^{(k+1)} = x^{(k)} + a_k g(x^{(k)})$, where a_k is taken to be as large as possible subject to $x^{(k+1)} \in \Lambda$. For some $K \leq n$, $g(x^{(K)}) = 0$. Then $x^{(K)}$ is the max-min fair allocation. We denote it by x^{maxmin} .

Equivalently, if the coordinates of $x^{\text{max-min}}$ and any other $x \in \Lambda$ are both reordered to be in nondecreasing order, then the reordered version of $x^{\text{max-min}}$ is lexicographically larger than the reordered version of x' .

Another characterization is that for any $x \in \Lambda$, if $x_r > x_r^{\text{max-min}}$ for some r , then there exists r' so that $x_{r'}' < x_{r'}^{\text{max-min}} \leq x_r^{\text{max-min}}$.

Still another characterization can be given for $x^{\text{max-min}}$, in the special case that $\Lambda = \{x \geq 0 : Ax \leq C\}$, as in the congestion control problem with hard link constraints. Setting $f^{\text{max-min}} = Ax^{\text{max-min}}$, the characterization is as follows: For each route r there is a link j_r so that $f_j^{\text{max-min}} = C_j$ and $x_r^{\text{max-min}} = \max\{x_{r'}^{\text{max-min}} : j \in r'\}$. Link j_r is called a *bottleneck* link for route r . In words, a bottleneck link for route r must be saturated, and no other user using link j_r can have a strictly larger allocation than user r . Due to this last characterization, a max-min fair allocation is sometimes called the *envy free* allocation, because for each user r there is a bottleneck link, and user r would not get a larger allocation by trading allocations with any of the other users using that link.

8.9 A Network Evacuation Problem

8.10 Braess Paradox

Consider the network shown in Fig. 8.5. The arrival process is assumed to be a Poisson process of rate λ . The upper branch of the network has a queue with a single exponential server followed by an infinite server queue with mean service time 2. The lower branch is similar, but with the order of the stations reversed. The traffic enters at a branch point, at which each customer is routed to one of the two branches, chosen independently and with equal probabilities. The scenario is equivalent to having a large number of independent Poisson arrival streams with sum of arrival rates equal to 2λ , in which all customers from some streams are routed to the upper branch, and all customers from the other streams are routed to the lower branch. For reasons that are apparent below, assume that $0 < \lambda < \mu - 1 < 2\lambda$. For example, $\lambda = 1.0$ and $\mu = 2.5$.

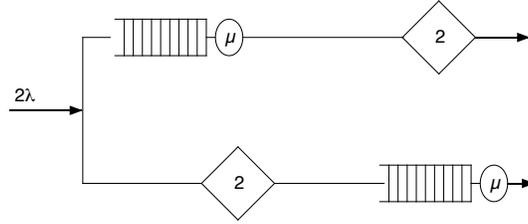


Figure 8.5: A network with two routes.

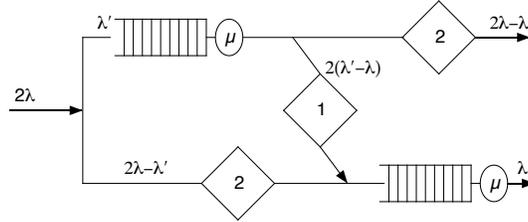


Figure 8.6: Augmented network.

Suppose that the choice of routing for each stream is made in a greedy fashion, so that no stream can improve its mean sojourn time by switching its routing choice. In game theory terminology, this is to say that the routing choices for the substreams form a Nash (or “person-by-person optimal”) equilibrium. By symmetry and by the convexity of the mean number of customers in an M/M/1 queue as a function of arrival rate, the Nash equilibrium is achieved when half the customers are routed to the upper branch, and half to the lower branch. This leads to Poisson arrival processes with parameter λ for each branch, and hence mean system time $D = 2 + 1/(\mu - \lambda)$ for both branches. The assumption that $\lambda < \mu - 1$ implies that $D < 3$.

Consider next the network of Fig. 8.6, which is obtained by augmenting the original network by a link with an infinite server queue with mean service time one. Let us consider what the new Nash equilibrium routing selection is. The augmented network has three routes: upper branch, lower branch, and cross-over route. Seek a solution in which all three routes are used by a positive fraction of the customers. This entails that the mean sojourn time for each of the three routes is the same. Let λ' denote the sum of the rates for the upper branch and cross-over routes. Equating the mean sojourn time of the cross-over route to that of the lower branch route yields that the mean system time, $1/(\mu - \lambda')$, of the single server station on the upper branch, should be 1. We thus set $\lambda' = \mu - 1$. Our assumptions on λ and μ insure that $\lambda' \in (0, 2\lambda)$, as desired. The mean sojourn time on the upper branch and lower branch should also be the same, so that the mean time in the single server system on the lower branch should also be one, so that its throughput should also be λ' . This determines the rates for all three routes, and the resulting traffic rates on the links of the network are shown in the figure. Further thought reveals this to be the *unique* Nash equilibrium routing selection.

Let us compare the mean sojourn time of customers in the two networks. The mean sojourn time in the original network is less than three, whereas in the augmented network it is equal to three. Thus, paradoxically, the mean sojourn time of all substreams is larger in the augmented

network than in the original network. Roughly speaking, the added link encourages customers to make heavier use of the single server systems, which are not as efficient from a social point of view. Of course if all streams of customers were to be banned from using the crossover link, then the performance observed in the original network would be obtained. This point corresponds to a *Pareto optimal* equilibrium point, and it clearly requires cooperation among the substreams.

8.11 Further reading and notes

The flow deviation algorithm was introduced in [15] and is discussed in [27], and is a special case of the Frank-Wolfe method. The flow deviation algorithms and other algorithms, based on use of second derivative information and projection, for congestion control and routing, are covered in [2]. For the decomposition of system problems into user and network problems, we follow [24]. Much additional material on the primal and dual algorithms, and second order convergence analysis based on the central limit theorem, is given in [25]. The notion of α -fairness is given in [31]. Additional material on algorithms and convergence analysis can be found in [38]. Braess discovered his paradox in a transportation network setting in 1968. It was adapted to the queueing context by Cohen and Kelly [6].

Today many subnetworks of the Internet use shortest path routing algorithms, but typically the routing is not dynamic, in that the link weights do not depend on congestion levels. Perhaps this is because the performance is deemed adequate, because there is plenty of capacity to spare. But it is also because of possible instabilities due to time lags. The recent surge in the use of wireless networks for high data rates has increased the use of dynamic routing (or load balancing) methods. Congestion control is central to the stability of the Internet, embodied by the transport control protocol (TCP) and van Jacobson's slow start method. The decomposition method discussed in this chapter matches well the congestion control mechanism used on the Internet today, and it gives insight into how performance could be improved.

8.12 Problems

8.1. A flow deviation problem (Frank-Wolfe method)

Consider the communication network with 24 links indicated in Figure 8.7. Each undirected

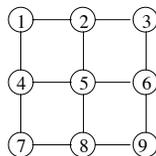


Figure 8.7: Mesh network with one source and one destination

link in the figure represents two directed links, one in each direction. There are four users: $S = \{(1, 9), (3, 7), (9, 1), (7, 3)\}$, each with demand b . The initial routing is deterministic, being concentrated on the four paths $(1, 2, 5, 6, 9), (3, 6, 5, 8, 7), (9, 8, 5, 4, 1)$, and $(7, 4, 5, 2, 3)$. Suppose that the cost associated with any one of the 24 links is given by $D_j(f) = f_j^2/2$, where f_j is the total flow on the link, measured in units of traffic per second.

- (a) What is the cost associated with the initial routing?
 (b) Describe the flow (for all four users) after one iteration of the flow deviation algorithm. Assume that, for any given user, any route from the origin to the destination can be used. Is the resulting flow optimal?

8.2. A simple routing problem in a queueing network

Consider the open queueing network with two customer classes shown in Figure 8.8. Customers of

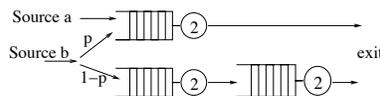


Figure 8.8: Open network with two types of customers

each type arrive according to a Poisson process with rate one. All three stations have a single exponential server with rate two, all service times are independent, and the service order is first-come, first-served. Each arrival of a customer of type b is routed to the upper branch with probability p , independently of the history of the system up to the time of arrival. (a) Find expressions for $D_a(p)$ and $D_b(p)$, the mean time in the network for type a and type b customers, respectively. (b) Sketch the set of possible operating points $\{(D_a(p), D_b(p)) : 0 \leq p \leq 1\}$. (c) Find the value p_a of p which minimizes $D_a(p)$. (d) Find the value p_b of p which minimizes $D_b(p)$. (e) Find the value p_{ave} of p which minimizes the average delay, $(D_a(p) + D_b(p))/2$. Also, how many iterations are required for the flow deviation algorithm (Frank-Wolfe method) to find p_{ave} ? (f) Find the value p_m of p which minimizes $\max\{D_a(p), D_b(p)\}$.

8.3. A joint routing and congestion control problem

Consider the network shown in Figure 8.9 and suppose (1,4) and (2,3) are the only two origin-

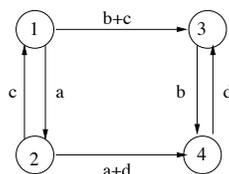


Figure 8.9: Network with joint routing and congestion control

destination pairs to be supported by the network. Suppose that the system cost for flow F on any link l is $D_l(F) = \frac{F^2}{2}$, and that the utility of flow r_{ij} for origin-destination pair (i, j) is $\beta_{ij} \log(r_{ij})$ for constants $\beta_{14}, \beta_{23} > 0$. Suppose that for each of the two origin-destination pairs that flow can be split between two paths. For convenience of notation we use the letters a, b, c, d to denote the values of the four path flows: $a = x_{124}$, $b = x_{134}$, $c = x_{213}$, and $d = x_{243}$. The joint routing and congestion control problem is to minimize the sum of link costs minus the sum of the utilities:

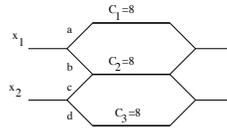
$$\min_{a,b,c,d \geq 0} \frac{1}{2}(a^2 + b^2 + c^2 + d^2 + (b+c)^2 + (a+c)^2) - \beta_{14} \log(a+b) - \beta_{23} \log(c+d)$$

- (a) Write out the optimality conditions. Include the possibility that some flow values may be

zero. (b) Show that if all flow values are positive, then $a = b$ and $c = d$. (c) Find the optimal flows if $\beta_{14} = 66$ and $\beta_{23} = 130$. Verify that for each route used, the price of the route (sum of $D'(F_l)$ along the route) is equal to the marginal utility of flow for the origin-destination of the route.

8.4. Joint routing and congestion control with hard link constraints

Consider the joint routing and congestion control problem for two users, each of which can split flow over two routes, as shown in the figure.



The total flow of user one is $x_1 = a + b$ and the total flow of user 2 is $x_2 = c + d$. The three central links each have capacity 8, and all other links have much larger capacity. User 1 derives value $U_1(a + b) = 2\sqrt{a + b}$ and user 2 derives value $U_2(c + d) = \ln(c + d)$. Let variables p_1, p_2, p_3 denote prices for the three links (i.e. Lagrange multipliers for the capacity constraints).

- Write down the optimality conditions for $a, b, c, d, p_1, p_2, p_3$.
- Identify the optimal assignment (a, b, c, d) and the corresponding three link prices.

8.5. Sufficiency of the optimality condition for hard link constraints

Consider a network with link set L , users indexed by a set of fixed routes R , and hard capacity constraints. Use the following notation:

- $A = (A_{lr} : l \in L, r \in R)$ with $A_{lr} = 1$ if link l is in route r , and $A_{lr} = 0$ otherwise
- $x = (x_r : r \in R)$ is the vector of flows on routes
- $f = (f_l : l \in L)$ is the vector of flows on links, given by $f = Ax$
- $C = (C_l : l \in L)$ is the vector of link capacities, assumed finite
- $U_r(x_r)$ is the utility function for route r , assumed concave, continuously differentiable, non-decreasing on $[0, +\infty)$.

(For simplicity routing is not considered.) The system optimization problem is $\max_{x: x \geq 0, Ax \leq C} \sum_r U_r(x_r)$. Suppose x^* is feasible ($x^* \geq 0$ and $Ax^* \leq C$) and satisfies the following condition (with $f^* = Ax^*$): There exists a vector $p = (p_l)$ for each link such that

$$p_l \geq 0, \quad \text{for } l \in L, \text{ with equality if } f_l^* < C_l$$

$$U'(x_r^*) \leq \sum_{l \in r} p_l \quad \text{for } r \in R, \text{ with equality if } x_r^* > 0$$

In this problem you are to prove that x^* is a solution to the system optimization problem.

- Define the Lagrangian

$$L(x, p) = \sum_r U_r(x_r) + \sum_l p_l \left(C_l - \sum_{r: l \in r} x_r \right)$$

Show that $L(x^*, p) \geq L(x, p)$ for any other vector x with $x \geq 0$. (Hint: There is no capacity constraint, and $L(x, p)$ is concave in x .) (b) Deduce from (a) that x^* is a solution to the system optimization problem. (Note: Since the feasible set is compact and the objective function is continuous, a maximizer x^* exists. Since the objective function is continuously differentiable and concave and the constraints linear, a price vector p satisfying the above conditions also exists, by a standard result in nonlinear optimization theory based on Farkas' lemma.)

8.6. Fair flow allocation with hard constrained links

Consider four flows in a network of three links as shown in Figure 8.10. Assume the capacity of

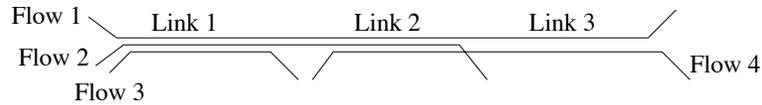


Figure 8.10: Network with four flows

each link is one. (a) Find the max-min fair allocation of flows. (b) Find the proportionally fair allocation of flows, assuming the flows are equally weighted. This allocation maximizes $\sum_i \log(x_i)$, where x_i is the rate of the i th flow. Indicate the corresponding link prices.

Chapter 9

Dynamic Network Control

9.1 Dynamic programming

The following game illustrates the concept of dynamic programming. Initially there are ten spots on a blackboard and a pair of ordinary dice is available. The player, a student volunteer from the audience, rolls the dice. Let i denote the number rolled on the first die and j denote the number rolled on the second die. The player then must either erase i spots, or j spots from the board. Play continues in a similar way. The player wins if at some time the number rolled on one of the dice is equal to the number of spots remaining on the board, so that all spots can be erased from the board. The player loses if at some time both numbers rolled are greater than the number of spots remaining on the board. For example, suppose the player begins by rolling (4,5) and elects to erase 4 and leave 6 spots on the board. Suppose the player next rolls (5,2) and elects to erase 2 more and leave 4 spots on the board. Suppose the player next rolls (4,2). The player wins because the four spots can be erased, leaving zero behind.

What is the optimal strategy for this game? What is the probability of winning for the optimal strategy? These questions are easily answered using dynamic programming. Fundamentally, dynamic programming is about working backwards from the end of a game. In the example at hand, the strategy is trivial if there is only one spot remaining on the board. Indeed, given one spot remains, the player wins if and only if one or both of the numbers rolled on the dice is 1, which has probability $w_1 = \frac{11}{36}$. The player has no choice. If there are only two spots remaining, the player wins if one of the numbers rolled is a two (happens with probability $\frac{11}{36}$). If one of the numbers is a one but neither is a two (happens with probability $\frac{9}{36}$), then the player erases one spot and will go on to win with probability $\frac{11}{36}$, because there will be one spot left. Thus, given the player has two spots left, the probability of winning the game is $w_2 = \frac{11}{36} + \frac{9}{36} * \frac{11}{36}$. Again, there is essentially no strategy needed. What should the player do if there are three spots left and the player rolls (1,2)? Since $w_2 > w_1$, clearly it is optimal for the player to erase just one spot, leaving two on the board.

In general, let w_n denote the probability of winning, for the optimal strategy, given there are n spots left on the board. Let $w_0 = 1$ and $w_k = 0$ for $k < 0$. If there are n spots left and (i, j) is rolled, the player has the choice of leaving $n - i$ spots or $n - j$ spots remaining on the board. Clearly the optimal strategy is to make the choice to maximize the probability of winning. This leads to the equation,

$$w_k = \frac{1}{36} \sum_{i=1}^6 \sum_{j=1}^6 \max\{w_{k-i}, w_{k-j}\} \quad (9.1)$$

Table 9.1: Probability of winning given n spots remaining

n	w_n
1	0.305556
2	0.381944
3	0.460455
4	0.537375
5	0.607943
6	0.666299
7	0.564335
8	0.588925
9	0.606042
10	0.617169

which recursively determines all the w_n 's. Numerical results are shown in Table 9.1.

In particular, given ten spots initially, the probability of winning the game is 0.617169. To compactly describe the optimal strategy, we list the numbers 1 through 9 according to decreasing values of the w_n 's: 6, 5, 9, 8, 7, 4, 3, 2, 1. Each time the player is faced with a decision, the optimal choice is the one that makes the number of spots remaining on the board appear earliest on the list. For example, if the first roll is (4,5), the player should erase 4 spots, because 6 appears on the list before 5.

Note that the optimal policy is deterministic and that decisions are based only on the remaining number of spots. There is no advantage for the player to randomize decisions or to recall the past history of the game. Note also that while the dynamic program equation (9.1) was derived by conditioning on the outcomes of one step forward in time, the equation is *solved* essentially by working backward from the end of the game. Working backwards from the end is the essence of dynamic programming.

9.2 Dynamic Programming Formulation

Dynamic control with complete state information is studied using the dynamic programming approach. The setting is pure jump controlled semi-Markov processes with independent and identically distributed holding times. The following are assumed given:

- A finite or countably infinite state space \mathcal{S} ,
- A real-valued function g on \mathcal{S} , denoting instantaneous cost,
- An action space \mathcal{U}
- A transition probability matrix $P(u) = (p_{ij}(u))_{i,j \in \mathcal{S}}$ for each $u \in \mathcal{U}$.
- A probability distribution function F , giving the distribution of inter-event times.

Definition 9.2.1 *A random process $(X(t) : t \geq 0)$ is a controlled semi-Markov process with control policy $\pi = (w_0, w_1, w_2, \dots)$ if*

- (a) $(X(t) : t \geq 0)$ has pure-jump sample paths.
- (b) There is a sequence of random variables $t_0 = 0, t_1, t_2, \dots$ such that $(t_{k+1} - t_k)_{k \geq 0}$ are independent with common distribution function F . The jump times of $(X(t) : t \geq 0)$ are all included in $\{t_0, t_1, \dots\}$
- (c) w_k is a function of $(X(t_j) : 0 \leq j \leq k)$ for each k
- (d) $P[X(t_{k+1}) = j \mid X(t_k) = i, w_k = u, (X(s) : s \leq t_k)] = p_{ij}(u)$
- (e) $(X(t_k) : k \geq 0)$ is independent of $(t_k : k \geq 0)$

Notes: Condition (c) is a causality condition on the control policy π , while condition (d) is a “semi-Markov” property. Conditions (a)-(e) can be greatly relaxed in what follows.

A cost function is used to evaluate a particular control policy π . A possible cost function is the discounted average cost

$$E_x \int_0^T g(X(t)) e^{-\alpha t} dt,$$

where $e^{-\alpha t}$ is a discount factor for inflation rate $\alpha > 0$, $T \in (0, \infty]$ is the time horizon, and “ E_x ” denotes expectation for initial state $x(0) = x$. Another cost function is the long term average cost:

$$\limsup_{T \rightarrow \infty} \frac{1}{T} E_x \int_0^T g(X(t)) dt$$

In order to apply the dynamic programming approach in a simple setting, it is useful to transfer attention to an equivalent discrete time control problem. The main idea is that

$$E_x \int_0^{t_n} g(X(t)) e^{-\alpha t} dt = \begin{cases} \frac{1-\beta}{\alpha} E_x \sum_{k=0}^{n-1} \beta^k g(X(t_k)) & \text{if } \alpha > 0 \\ \bar{t}_1 E_x \sum_{k=0}^{n-1} g(X(t_k)) & \text{if } \alpha = 0 \end{cases}$$

where $\beta = \hat{F}(\alpha) = \int_0^\infty e^{-\alpha s} dF(s)$. Thus, control of the discrete time process $(X(t_k))$ is equivalent to control of the original process $(X(t))$. Often we will write $X(k)$ for $X(t_k)$.

Example (Flow control — an $M_{\text{controlled}}/M/1$ queue) Consider a system consisting of a queue and a server. Let $X(t)$ denote the number of customers in the system. Suppose customers arrive at instantaneous rate λu , where u denotes a control variable with $u \in [0, 1]$. Thus, λ is the maximum possible arrival rate, and the effect of the control is to select a possibly smaller actual arrival rate. The server is an exponential type server with rate μ . Thus, the parameters of the system are the maximum arrival rate λ , and the service rate μ . By taking the control u to be a constant, the model reduces to an M/M/1 system with arrival rate λu . In this example the control will be allowed to depend on the number in the system.

Roughly speaking, the control strategy should insure a large throughput, while keeping the number in the system small. Of course, setting $u \equiv 1$ maximizes the expected throughput, but, especially if $\lambda > \mu$, this can lead to a large number in the system. On the other hand, if $u \equiv 0$ then the number in the system is minimized, but the long run throughput is zero. We will thus select a cost function which reflects both customer holding cost and a reward for throughput.

This example can be fit into the dynamic programming framework by choosing $\mathcal{S} = \{0, 1, 2, \dots\}$, $\mathcal{U}=[0,1]$,

$$P(u) = \begin{bmatrix} \frac{\mu+\lambda(1-u)}{\gamma} & \frac{\lambda u}{\gamma} & 0 & 0 \\ \frac{\mu}{\gamma} & \frac{\lambda(1-u)}{\gamma} & \frac{\lambda u}{\gamma} & 0 \\ 0 & \frac{\mu}{\gamma} & \frac{\lambda(1-u)}{\gamma} & \ddots \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & & \end{bmatrix},$$

and taking F to be the exponential distribution function with parameter $\gamma = \lambda + \mu$.

A reasonable objective is to minimize the discounted average cost,

$$cE_x \int_0^T e^{-\alpha t} X(t) dt - E_x \sum_{t \leq T} I_{\{\Delta X(t)=-1\}} e^{-\alpha t}, \quad (9.2)$$

where $\alpha > 0$ is a discount factor, and $c > 0$ is a constant. The integral term in (9.2) reflects holding cost c per unit time per customer in the system, and the sum term in (9.2) reflects a unit reward per departure. Replacing the departure process by its compensator (integrated intensity), the cost in (9.2) can be reexpressed as

$$E_x \int_0^T e^{-\alpha t} g(X(t)) dt - \underbrace{\mu E_x \int_0^T e^{-\alpha t} dt}_{\text{policy independent}} \quad (9.3)$$

where $g(x) = cx + \mu I_{\{x=0\}}$. Since the second term in (9.3) does not depend on the control policy used, the term can be dropped from the cost function. Thus, the function g taken in the dynamic programming formulation is $g(x) = cx + \mu I_{\{x=0\}}$. Intuitively, the cost $g(0) = \mu$ accounts for lost departure opportunities when the system is empty, whereas the linear term cx gives the holding cost of customers in the system.

9.3 The Dynamic Programming Optimality Equations

The discrete time problem is to choose π to minimize $E_x \sum_{k=0}^{N-1} \beta^k g(X(k))$.

Define

$$V_n(x) = \inf_{\pi} E_x \sum_{k=0}^{n-1} \beta^k g(X(k)).$$

The value $V_n(x)$ is the minimum cost if n terms are included in the cost function, given initial state x . Set by convention $V_0(x) = 0$, and observe that $V_1(x) = g(x)$. The fundamental backwards equation of dynamic programming is

$$V_{n+1}(x) = g(x) + \beta \inf_{u \in \mathcal{U}} \sum_y p_{xy}(u) V_n(y), \quad (9.4)$$

for $n \geq 0$. Furthermore, a policy π is optimal if the control value when there are n steps to go is a function $u_n(x)$ of the present state, where the feedback control function u_n is given by

$$u_n(x) = \operatorname{argmin}_{u \in \mathcal{U}} \sum_y p_{xy}(u) V_n(y) \quad (9.5)$$

Together (9.4) and (9.5) determine the optimal control values. If the state space is finite, these equations provide a numerical method for computation of the optimal policies. For very large or infinite state spaces, some progress can often be made by approximating the process by one with a moderate number of states and again seeking numerical results. Still another use of the equations is to verify structural properties of optimal feedback control functions u_n (such as a “threshold” or “switching curve” structure). These properties can serve to restrict the search for optimal policies to a much smaller set of policies. An approach to verifying structural properties for the u_n is to seek a *set* of properties for the cost-to-go functions V_n : (a) which can be proved by induction on n via (9.4), and (b) which imply the structural properties of the u_n via (9.5).

Return to Example To illustrate the method, let us return to the $M_{\text{controlled}}/M/1$ example, and prove the following claim: An optimal decision for n steps to go is given by

$$u_n(x) = \begin{cases} 1 & \text{if } x \leq s(n) \\ 0 & \text{if } x > s(n) \end{cases} \quad (9.6)$$

for some threshold value $s(n)$.

The dynamic programming recursion (9.4) for this example becomes

$$V_{n+1}(x) = cx + \mu I_{\{x=0\}} + \beta \min_{u \in [0,1]} \left\{ \frac{\mu}{\gamma} V_n((x-1)) + \frac{\lambda(1-u)}{\gamma} V_n(x) + \frac{\lambda u}{\gamma} V_n(x+1) \right\} \quad (9.7)$$

and (9.5) becomes

$$u_n(x) = \begin{cases} 0 & \text{if } V_n(x) < V_n(x+1) \\ 1 & \text{if } V_n(x) > V_n(x+1) \\ \text{arbitrary} & \text{if } V_n(x) = V_n(x+1) \end{cases} \quad (9.8)$$

Equation (9.8) shows that without loss of optimality, the control values can be restricted to values 0 and 1. Furthermore, (9.7) simplifies to

$$V_{n+1}(x) = cx + \mu I_{\{x=0\}} + \frac{\beta\mu}{\gamma} V_n((x-1)_+) + \frac{\beta\lambda}{\gamma} \min\{V_n(x), V_n(x+1)\} \quad (9.9)$$

In view of (9.8), the claim is implied by the following property of the V_n :

$$V_n(x+1) - V_n(x) \begin{cases} \leq 0 & \text{if } x \leq s(n) \\ \geq 0 & \text{if } x > s(n) \end{cases} \quad (9.10)$$

for some threshold value $s(n)$. While (9.10) implies that the structure (9.6) is optimal, the property (9.10) does not follow by induction from (9.9). Thus, a *stronger* property is sought for V_n which can be proved by induction.

The following two properties can be proved jointly by induction on n :

(a) $V_n(x)$ is convex (i.e. $V_n(x+1) - V_n(x)$ is nondecreasing in x)

(b) $V_n(1) - V_n(0) \geq -\gamma/\beta$.

Also, property (a) implies (9.10) for some value of the threshold $s(n)$, and hence establishes that the threshold structure for u_n is optimal.

Proof. Here is the induction argument. The base case, that V_0 satisfies (a) and (b), is immediate. Suppose that V_n satisfies properties (a) and (b). It must be shown that V_{n+1} has properties

(a) and (b). Define $V_n(-1) = V_n(0) + (\gamma/\beta)$. By property (b) for V_n , the new function V_n is convex on the set $\{-1, 0, 1, \dots\}$, and (9.9) becomes

$$V_{n+1}(x) = cx + \frac{\beta\mu}{\gamma}V_n((x-1)_+) + \frac{\beta\lambda}{\gamma} \min\{V_n(x), V_n(x+1)\}, \quad (9.11)$$

for $x \geq 0$. Since each of the three terms on the right of (9.11) is convex on $\{0, 1, 2, \dots\}$ (sketch an example for the last of the three terms) it follows that V_{n+1} satisfies property (a). ■

Note that property (b) of V_n was used to prove property (a) for V_{n+1} . In fact, that is the only reason we chose to introduce property (b). It remains to verify property (b) for V_{n+1} . That is accomplished by the following string of inequalities, where “ $a \wedge b$ ” denotes the minimum of two numbers a and b :

$$\begin{aligned} V_{n+1}(1) - V_{n+1}(0) &= -\mu + c + \frac{\beta\lambda}{\gamma}[(V_n(2) \wedge V_n(1)) - (V_n(1) \wedge V_n(0))] \\ &\geq -\mu + \frac{\beta\lambda}{\gamma}[(V_n(2) \wedge V_n(1)) - V_n(1)] \\ &= -\mu + \frac{\beta\lambda}{\gamma}(V_n(2) - V_n(1)) \wedge 0 \\ &\geq -\mu + \frac{\beta\lambda}{\gamma}(V_n(1) - V_n(0)) \wedge 0 \\ &\geq -\mu + \frac{\beta\lambda}{\gamma}\left(-\frac{\gamma}{\beta}\right) = -\gamma \geq -\gamma/\beta. \end{aligned}$$

The proof that V_{n+1} , and hence all the functions V_n , have properties (a) and (b), is proved.

Some remarks concerning extensions of the above result are in order. The threshold values $s(n)$ depend on n . However, it can be shown that $\lim_{n \rightarrow \infty} V_n(x)$ exists and is finite for each x . Call the limit function $V_\infty(x)$. It can also be shown that the limit V_∞ determines the optimal strategies for the infinite horizon, discounted cost problem. Since properties (a) and (b) are inherited by the limit, conclude that there is an optimal control strategy for the infinite horizon problem given by a fixed threshold s : new arrivals are allowed if and only if the number in the system is less than s . Moreover, unless it happens that $V_\infty(x) = V_\infty(x+1)$ for some value of x , the threshold policy is the unique optimal control. It can also be shown that $\lim_{\beta \rightarrow 1} [V_\infty(x) - V_\infty(0)]$ exists and gives rise to the control for the long-term average cost criterion. The limit is again convex, so that the long-term average cost is also minimized by a fixed threshold policy.

Example (Dynamic routing to two exponential server queues)

Consider a system with two service stations, each consisting of a queue plus server, as shown in Figure 9.1. Customers arrive according to a Poisson process of rate $\lambda > 0$. A customer is routed to the first station with probability u , and to the second station with probability $1 - u$, where u is a variable to be controlled, and can depend on the state of the system. Service times at the first station are exponentially distributed with parameter μ_1 , and service times at the second station are exponentially distributed with parameter μ_2 . The holding cost per unit time per customer in station 1 is c_1 and in station 2 is c_2 , where $c_i > 0$ for $i = 1, 2$. This example fits into the dynamic programming framework we have described, with the following choices:

$$U = [0, 1]$$

$\mathcal{S} = \mathbb{Z}_+^2$, where state $x = (x_1, x_2)$ denotes x_i customers in station i , for $i = 1, 2$.

$$g(x) = c_1 x_1 + c_2 x_2$$

F corresponds to the exponential distribution with parameter $\gamma = \lambda + \mu_1 + \mu_2$.

$$p_{x,y}(u) = \frac{1}{\gamma} (\mu_1 I_{\{y=D_1x\}} + \mu_2 I_{\{y=D_2x\}} + u\lambda I_{\{y=A_1x\}} + (1-u)\lambda I_{\{y=A_2x\}}) \text{ where}$$

$$A_1x = (x_1 + 1, x_2), A_2x = (x_1, x_2 + 1), D_1x = ((x_1 - 1)_+, x_2), \text{ and } D_2x = (x_1, (x_2 - 1)_+).$$

The backwards equation of dynamic programming becomes

$$V_{n+1}(x) = g(x) + \min_{0 \leq u \leq 1} \frac{\beta}{\gamma} \{ \mu_1 V_n(D_1x) + \mu_2 V_n(D_2x) + u\lambda V_n(A_1x) + (1-u)\lambda V_n(A_2x) \}$$

or, after plugging in the optimal value for u ,

$$V_{n+1}(x) = g(x) + \frac{\beta}{\gamma} \{ \mu_1 V_n(D_1x) + \mu_2 V_n(D_2x) + \lambda \min\{V_n(A_1x), V_n(A_2x)\}, \}$$

with the initial condition $V_0 \equiv 0$. Furthermore, the optimal control is given in feedback form as follows. Given there are n steps to go and the state is x , the optimal control action is $u_n^*(x)$, given by

$$u_n^*(x) = \begin{cases} 1 & \text{if } V_n(A_1x) \leq V_n(A_2x) \\ 0 & \text{else.} \end{cases} \quad (9.12)$$

That is, if the current state is x and an arrival occurs, the optimal control routes the arrival to whichever station yields the lower expected cost.

Consider the symmetric case: $\mu_1 = \mu_2$ and $c_1 = c_2$. We will prove that $u_n^*(x) = I_{\{x_1 \leq x_2\}}$ for all $n \geq 1$. That is, the send to shorter policy is optimal. It is easy to check that the control specification (10.20) is equivalent to $u_n^*(x) = I_{\{x_1 \leq x_2\}}$, if V_n has the following three properties:

1. (symmetry) $V(x_1, x_2) = V(x_2, x_1)$ for all $(x_1, x_2) \in \mathcal{S}$
2. (monotonicity) $V(x_1, x_2)$ is nondecreasing in x_1 and in x_2 .
3. $V(x_1, x_2) \leq V(x_1 - 1, x_2 + 1)$ whenever $0 \leq x_1 \leq x_2$.

Together properties 1 and 3 mean that when restricted to the states along a line segment of the form $\{x \in \mathcal{S} : x_1 + x_2 = l\}$ for $l \geq 1$, the cost function is a monotonically nondecreasing function of the distance from the midpoint.

Thus, it remains to prove that V_n has the three properties stated, for all $n \geq 0$. This is done by induction. Trivially, the function V_0 satisfies all three properties. So for the sake of argument by induction, suppose that V_n satisfies all three properties. We show that V_{n+1} satisfies all three

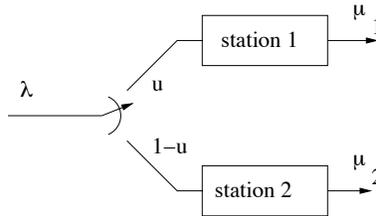


Figure 9.1: Dynamic routing to two service stations

properties. Property 1 for V_{n+1} follows from property 1 for V_n and the symmetry of the backwards equations in x_1 and x_2 . Property 2 for V_{n+1} follows from property 2 for V_n . It remains to show that V_{n+1} has property 3. Since property 3 is closed under summation, it suffices to prove that $g(x)$, $V_n(D_1x)$, $V_n(D_2x)$, and $\min\{V_n(A_1x), V_n(A_2x)\}$ have property 3. It is easily checked that the first three of these functions has property 3, so it remains to prove that $\min\{V_n(A_1x), V_n(A_2x)\}$ has property 3. So suppose that $1 \leq x_1 \leq x_2$, and refer to Figure 9.2. It must be shown that

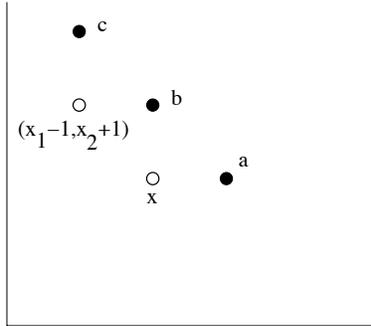


Figure 9.2: V_n is sampled to get values a , b , and c .

$a \wedge b \leq b \wedge c$ where $a = V_n(x_1 + 1, x_2)$, $b = V_n(x_1, x_2 + 1)$, and $c = V_n(x_1 - 1, x_2 + 2)$. We argue first that $a \leq b$, by considering two cases: If $x_1 = x_2$ then the states $(x_1, x_2 + 1)$ and $(x_1 + 1, x_2)$ are obtained from each other by swapping coordinates, so that $a = b$ by property 1 for V_n . If $x_1 > x_2$ then $x_1 \geq x_2 + 1$ so that $a \leq b$ by property 3 for V_n . Thus, in general $a \leq b$. Similarly, $b \leq c$ by property 3 for V_n . Thus, $a \leq b \leq c$, which immediately implies $a \wedge b \leq b \wedge c$, as desired. The proof of the induction step is complete. Therefore, V_n has properties 1-3 for all $n \geq 0$, and the send to shorter queue policy is optimal, in the symmetric case.

In the general case, the optimal control is given by $u_n^*(x) = I_{\{x_2 > s_n(x_1)\}}$, where s_n is a nondecreasing function. This fact can be established by proving, by induction on n , that the cost-to-go functions V_n have a more elaborate set of properties [19].

9.4 Problems

9.1. Illustration of dynamic programming – a stopping time problem

Let $n \geq 1$ and let X_1, \dots, X_n be mutually independent, exponentially distributed random variables with mean one. A player observes the values of the random variables one at a time. After each observation the player decides whether to stop making observations or to continue making observations. The player's score for the game is the last value observed. Let V_n denote the expected score for an optimal policy. Note that $V_1 = 1$. (a) Express V_{n+1} as a function of V_n . (Hint: Condition on the value of X_1 .) (b) Describe the optimal policy for arbitrary value n . (c) For $1 \leq n \leq 5$, compare V_n to $E[\max(X_1, \dots, X_n)] = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, which is the maximum expected score in case the player learns the values of the X 's before play begins.

9.2. Comparison of four ways to share two servers

Let N_i denote the mean total number of customers in system i in equilibrium, where systems 1 through 4 are shown. The arrival process for each system is Poisson with rate 2λ and the servers

are exponential with parameter μ . (a) Compute N_i as a function of $\rho = \lambda/\mu$ for $1 \leq i \leq 3$. (Let

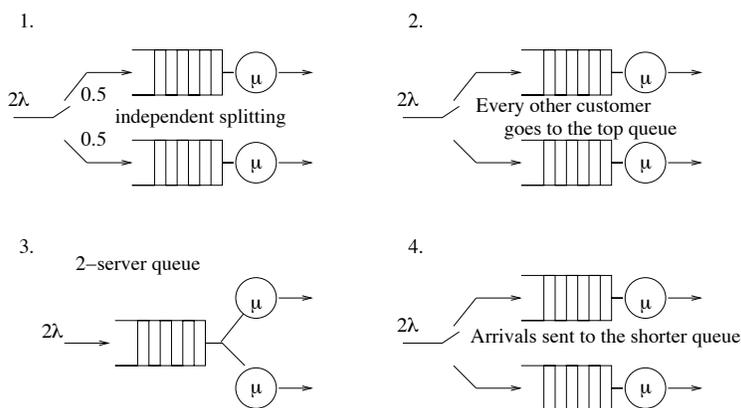


Figure 9.3: Four two-server systems

me know if you can find N_4 .) (b) Order N_1, N_2, N_3, N_4 from smallest to largest. Justify answer. Does your answer depend on ρ ?

9.3. A stochastic dynamic congestion control problem

Consider a single server queue with finite waiting room so that the system can hold at most K customers. The service times are independent and exponential with parameter μ . If a customer arrives and finds the system full then the customer is rejected and a penalty c is assessed. For each customer that is served a reward r is received at the time of the customer's departure. There are two sources of customers. One is Poisson with fixed rate λ_o and the other is a variable rate source with instantaneous rate λu , where λ is a constant and u is the control value, assumed to lie in the interval $[0,1]$. The control value u can depend on the state.

(a) Formulate a dynamic programming problem. In particular, describe the state space, the instantaneous cost function, the distribution of inter-event times, and the transition matrix $P(u)$ (draw the transition probability diagram). (b) Give the equations for the cost-to-go functions V_n . (Assume a discount factor $\beta < 1$ for the discrete time formulation as in class.) (c) Describe the optimal control law explicitly in terms of the cost-to-go functions. (d) Speculate about the structure of optimal policy. Can you prove your conjecture? (e) Suppose the assumptions were modified so that customers from the first source are served at rate μ_1 and that customers from the second are served at rate μ_2 . Describe the state space that would be needed for a controlled Markov description of the system under (i) FCFS service order, (ii) preemptive resume priority to customer from the first source, or (iii) processor sharing.

9.4. Conversion to discrete time with control dependent cost

The continuous time control problem can still be converted to a discrete time control problem if the cost per unit time function g depends on u because, in that case,

$$E_x \int_0^{t_n} g(X(t), u(t)) e^{-\alpha t} dt = \begin{cases} \frac{1-\beta}{\alpha} E_x \sum_{k=0}^{n-1} \beta^k g(X(t_k), w_k) & \text{if } \alpha > 0 \\ \bar{t}_1 E_x \sum_{k=0}^n g(X(t_k), w_k) & \text{if } \alpha = 0 \end{cases} \quad (9.13)$$

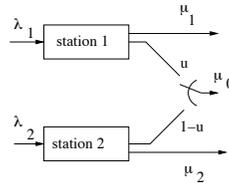
where $\beta = \hat{F}(\alpha) = \int_0^\infty e^{-\alpha s} dF(s)$. (a) Prove (9.13) in case $\alpha > 0$. (The proof for $\alpha = 0$ is sim-

ilar.) Thus, by ignoring constant factors, we can take the cost for n terms in discrete time to be $E_x \sum_{k=0}^{n-1} g(X(t_k), w_k)$.

(b) How should the fundamental backwards recursion of dynamic programming be modified to take into account the dependence of g on u ?

9.5. Optimal control of a server

Consider a system of two service stations as pictured.



Station i has Poisson arrivals at rate λ_i and an exponential type server, with rate $m_i(u)$, where $m_1(u) = \mu_1 + u\mu_0$ and $m_2(u) = \mu_2 + (1-u)\mu_0$ and u is a control variable with $u \in U = [0, 1]$. Suppose we are interested in the infinite horizon discounted average number of customers in the system, with discount rate $\alpha > 0$.

(a) Specify the transition probability matrix, $P(u)$, the interjump CDF, F , and cost function g , for a dynamic programming model.

(b) Write the dynamic programming update equations for the optimal cost-to-go functions, V_n .

(c) Express the optimal state feedback control for n -steps, $u_n^*(x)$, in terms of V_n .

(d) Consider the symmetric case: $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$. Give an educated guess regarding the structure of the optimal control, and give an equivalent condition on V_n . (Note: An equivalent condition, not just a sufficient condition, is asked for here.)

9.6. A dynamic server rate control problem with switching costs

Consider the following variation of an M/M/1 queueing system. Customers arrive according to a Poisson process with rate λ . The server can be in one of two states—the low state or the high state. In addition, the server is either busy or idle, depending on whether there are customers in the system. A control policy is to be found which determines when the server should switch states, either from low to high or from high to low. The costs involved are a cost per unit time for each customer waiting in the system, a cost per unit time of having the server in the high state, and a switching cost, assessed each time that the server is switched from the low state to the high state. Specifically, let μ_H (respectively, μ_L) denote the service rate when the server is in the high state (respectively, low state), where $\mu_H > \mu_L > 0$. Let c_H denote the added cost per unit time for having the server in the high state, and let c_S denote the cost of switching the server from the low state to the high state. Assume that the control can switch the state of the server only just after a customer arrival or potential departure. Assume there is no charge for switching the server from the high state to the low state. Finally, let c_W denote the cost per unit time per customer waiting in the system. The controller has knowledge of the server state and the number of customers in the system. Suppose the goal is to minimize the expected cost over the infinite horizon, discounted at rate α . (a) Formulate the control problem as a semi-Markov stochastic control problem. In particular, indicate the state space you use, and the transition probabilities. (Hint: Use a control dependent cost function to incorporate the switching cost. The switching cost is incurred at discrete times, in the same way the rewards are given at discrete times for the $M_{\text{controlled}}/M/1$ example

in the notes, and can be handled similarly. Using a two dimensional control, one coordinate to be used in case of an arrival, and one coordinate to be used in case of a departure, is helpful.) (b) Let V_n denote the cost-to-go function when there are n steps in the equivalent discrete-time control problem. Write down the dynamic programming equation expressing V_{n+1} in terms of V_n and indicate how the optimal control for n steps to go depends on V_n . (c) Describe the qualitative behavior you expect for the optimal control for the infinite horizon problem. In particular, indicate a particular threshold or switching curve structure you expect the control to have. (d) Describe the optimal control policy when the following two conditions both hold: there is no switching cost (i.e. $c_S = 0$), and $c_W(\mu_H + \alpha) > (\mu_L + \alpha)(c_W + c_H)$. (Hint: Consider the expected cost incurred during the service of one customer.)

Chapter 10

Solutions

1.1. Poisson merger

Suppose X_i is a Poisson random variable with mean λ_i for $i \in \{1, 2\}$ such that X_1 and X_2 are independent. Let $X = X_1 + X_2$, $\lambda = \lambda_1 + \lambda_2$, and $p_i = \lambda_i/\lambda$. Then for $k \geq 0$,

$P[X = k] = \sum_{j=0}^k P[X_1 = j]P[X_2 = k - j] = \sum_{j=0}^k \frac{e^{-\lambda_1} \lambda_1^j}{j!} \frac{e^{-\lambda_2} \lambda_2^{k-j}}{(k-j)!} = \frac{e^{-\lambda} \lambda^k}{k!} [\sum_{j=0}^k \binom{k}{j} p_1^j p_2^{k-j}] = \frac{e^{-\lambda} \lambda^k}{k!}$, where we use the fact that the sum in square brackets is the sum over the binomial distribution with parameters k and p_1 . Thus, the sum of two independent Poisson random variables is a Poisson random variable. By induction this implies that the sum of any number of independent Poisson random variables is also Poisson.

Now suppose that $N_j = (N_j(t) : t \geq 0)$ is a Poisson process with rate λ_j for $1 \leq j \leq K$, and that N_1, \dots, N_K are mutually independent. Given $t_0 < t_1 < \dots < t_p$, consider the following array of random variables:

$$\begin{array}{ccccccc}
 N_1(t_1) - N_1(t_0) & N_1(t_2) - N_1(t_1) & \cdots & N_1(t_p) - N_1(t_{p-1}) & & & \\
 N_2(t_1) - N_2(t_0) & N_2(t_2) - N_2(t_1) & \cdots & N_2(t_p) - N_2(t_{p-1}) & & & \\
 \vdots & \vdots & \cdots & \vdots & & & \\
 N_K(t_1) - N_K(t_0) & N_K(t_2) - N_K(t_1) & \cdots & N_K(t_p) - N_K(t_{p-1}) & & &
 \end{array} \tag{10.1}$$

The rows of the array are independent by assumption, and the variables within a row of the array are independent and Poisson distributed by Proposition 1.5.1, characterization (iii). Thus, the elements of the array are mutually independent and each is Poisson distributed. Let $N(t) = N_1(t) + \dots + N_K(t)$ for all $t \geq 0$. Then the vector of random variables

$$N(t_1) - N(t_0) \quad N(t_2) - N(t_1) \quad \cdots \quad N(t_p) - N(t_{p-1}) \tag{10.2}$$

is obtained by summing the rows of the array. Therefore, the variables of the row are independent, and for any i the random variable $N(t_{i+1}) - N(t_i)$ has the $Poi(\lambda(t_{i+1} - t_i))$ distribution, where $\lambda = \lambda_1 + \dots + \lambda_K$. Thus, N is a rate λ Poisson process by Proposition 1.5.1, characterization (iii).

1.2. Poisson splitting

This is basically the first problem in reverse. Let X be Poisson random variable, and let each of X individuals be independently assigned a type, with type i having probability p_i , for some

probability distribution p_1, \dots, p_K . Let X_i denote the number assigned type i . Then,

$$\begin{aligned} P(X_1 = i_1, X_2 = i_2, \dots, X_K = i_K) &= P(X = i_1 + \dots + i_K) \binom{i_1 + \dots + i_K}{i_1! i_2! \dots i_K!} p_1^{i_1} \dots p_K^{i_K} \\ &= \prod_{j=1}^K \frac{e^{-\lambda_j} \lambda_j^{i_j}}{i_j!} \end{aligned}$$

where $\lambda_i = \lambda p_i$. Thus, independent splitting of a Poisson number of individuals yields that the number of each type i is Poisson, with mean $\lambda_i = \lambda p_i$ and they are independent of each other.

Now suppose that N is a rate λ process, and that N_i is the process of type i points, given independent splitting of N with split distribution p_1, \dots, p_K . Proposition 1.5.1, characterization (iii). the random variables in (10.2) are independent, with the i^{th} having the $Poi(\lambda(\lambda(t_{i+1} - t_i)))$ distribution. Suppose each column of the array (10.1) is obtained by independent splitting of the corresponding variable in (10.2). Then by the splitting property of random variables, we get that all elements of the array (10.1) are independent, with the appropriate means. By Proposition 1.5.1, characterization (iii), the i^{th} process N_i is a rate λp_i random process for each i , and because of the independence of the rows of the array, the K processes N_1, \dots, N_K are mutually independent.

1.3. Poisson method for coupon collector's problem

(a) In view of the previous problem, the number of coupons of a given type i that arrive by time t has the Poisson distribution with mean $\frac{t}{k}$, and the numbers of arrivals of different types are independent. Thus, at least one type i coupon arrives with probability $1 - e^{-t/K}$, and $p(k, t) = (1 - e^{-t/k})^k$.

(b) Therefore, $p(k, k \ln k + kc) = (1 - \frac{e^{-c}}{k})^k$, which, by the hint, converges to $e^{-e^{-c}}$ as $k \rightarrow \infty$.

(c) The increments of A over intervals of the form $[k, k+1]$ are independent, $Poi(1)$ random variables. Thus, the central limit theorem can be applied to $A(t)$ for large t , yielding that for any constant D , $\lim_{t \rightarrow \infty} P[A(t) \geq t + D\sqrt{t}] = Q(D)$, where Q is the complementary normal CDF. The problem has to do with deviations of size $(c - c')k$, which for any fixed D , grow to be larger than $D\sqrt{k \ln k + kc}$. Thus, with $\epsilon = |c - c'|$, $P[|A(k \ln k + kc) - (k \ln k + kc)| \geq \epsilon k] \leq Q(D)$ for k large enough, for any D . Thus, $\lim_{k \rightarrow \infty} P[|A(k \ln k + kc) - (k \ln k + kc)| \geq \epsilon k] = 0$, which is equivalent to the required fact.

(d) Let C_t be the event that the collection is complete at time t . Condition on the value of $A(t)$.

$$p(k, t) = P(C_t) = P(C_t | A(t) \geq n)P(A(t) \geq n) + P(C_t | A(t) < n)P(A(t) < n) \quad (10.3)$$

First we bound from below the right side of (10.3). Since getting more coupons can only help complete a collection, $P(C_t | A(t) \geq n) \geq P(C_t | A(t) = n) = d(k, n)$, and other terms on the righthand side of (10.3) are nonnegative. This yields the desired lower bound.

Similarly, to bound above the right side of (10.3) we use $P(C_t | A(t) \geq n) \leq 1$ and $P(C_t | A(t) < n) \leq d(k, n)$.

(e) Fix c . The first part of the inequality of part (c) yields that $d(k, n) \leq p(k, t)/P[A(t) \geq n]$. Suppose $n = k \ln k + kc$. Let $c' > c$ and take $t = k \ln k + kc'$ as $k \rightarrow \infty$. Then $p(k, t) \rightarrow e^{-e^{-c'}}$ by part (b) and $P[A(t) \geq n] \rightarrow 1$. Thus, $\limsup d(k, n) \leq e^{-e^{-c'}}$. Since c' is arbitrary with $c' > c$, $\limsup d(k, n) \leq e^{-e^{-c}}$. A similar argument shows that $\liminf d(k, n) \geq e^{-e^{-c}}$. Therefore, $\lim d(k, n) = e^{-e^{-c}}$.

1.4. The sum of a random number of random variables

(a)

$$\begin{aligned} E[S] &= \sum_{n=0}^{\infty} E[S|N=n]P[N=n] = \sum_{n=0}^{\infty} E[X_1 + \cdots + X_n|N=n]P[N=n] \\ &= \sum_{n=0}^{\infty} E[X_1 + \cdots + X_n]P[N=n] = \sum_{n=0}^{\infty} n\bar{X}P[N=n] = \bar{X}\bar{N}. \end{aligned}$$

Similarly,

$$\begin{aligned} E[S^2] &= \sum_{n=0}^{\infty} E[(X_1 + \cdots + X_n)^2]P[N=n] \\ &= \sum_{n=0}^{\infty} [E[X_1 + \cdots + X_n]^2 + \text{Var}(X_1 + \cdots + X_n)] P[N=n] \\ &= \sum_{n=0}^{\infty} [(n\bar{X})^2 + n\text{Var}(X)] P[N=n] \\ &= \bar{N}^2(\bar{X})^2 + \bar{N}\text{Var}(X) \end{aligned}$$

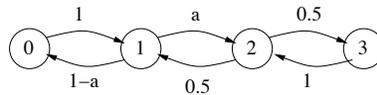
so that $\text{Var}(S) = E[S^2] - Es^2 = \text{Var}(N)(\bar{X})^2 + \bar{N}\text{Var}(X)$.

(b) By the same reasoning as in part (a),

$$\begin{aligned} \Phi_S(u) &= E[e^{juS}] = \sum_{n=0}^{\infty} E[e^{ju(X_1 + \cdots + X_n)}]P[N=n] \\ &= \sum_{n=0}^{\infty} \Phi_{X_1}(u)^n P[N=n] = B(\Phi_{X_1}(u)) \end{aligned}$$

1.5. Mean hitting time for a simple Markov process

(a)



Solve $\pi = \pi P$ and $\pi e = 1$ to get $\pi = (\frac{1-a}{2(1+a)}, \frac{1}{2(1+a)}, \frac{2a}{2(1+a)}, \frac{a}{2(1+a)})$ for all $a \in [0, 1]$ (unique).

(b) A general way to solve this is to let $h_i = E[\min\{n \geq 0 | X(n) = 3\} | X(0) = i]$, for $0 \leq i \leq 3$. Our goal is to find h_0 . Trivially, $h_3 = 0$. Derive equations for the other values by conditioning on

$$h_0 = 1 + h_1$$

the first step of the process: $h_i = 1 + \sum_j p_{ij}h_j$ for $i \neq 3$. Or $h_2 = 1 + (1-a)h_0 + ah_2$ yielding

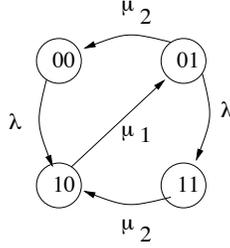
$$h_2 = 1 + (0.5)h_i$$

$$\begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{a} + 1 \\ \frac{4}{a} \\ \frac{2}{a} + 1 \end{pmatrix}. \text{ Thus, } h_0 = \frac{4}{a} + 1 \text{ is the required answer.}$$

1.6. A two station pipeline in continuous time

(a) $\mathcal{S} = \{00, 01, 10, 11\}$

(b)



$$(c) Q = \begin{pmatrix} -\lambda & 0 & \lambda & 0 \\ \mu_2 & -\mu_2 - \lambda & 0 & \lambda \\ 0 & \mu_1 & -\mu_1 & 0 \\ 0 & 0 & \mu_2 & -\mu_2 \end{pmatrix}.$$

(d) $\eta = (\pi_{00} + \pi_{01})\lambda = (\pi_{01} + \pi_{11})\mu_2 = \pi_{10}\mu_1$. If $\lambda = \mu_1 = \mu_2 = 1.0$ then $\pi = (0.2, 0.2, 0.4, 0.2)$ and $\eta = 0.4$.

(e) Let $\tau = \min\{t \geq 0 : X(t) = 00\}$, and define $h_s = E[\tau | X(0) = s]$, for $s \in \mathcal{S}$. We wish to find

$$h_{00} = 0$$

$$h_{01} = \frac{1}{\mu_2 + \lambda} + \frac{\mu_2 h_{00}}{\mu_2 + \lambda} + \frac{\lambda h_{11}}{\mu_2 + \lambda}$$

$$h_{10} = \frac{1}{\mu_1} + h_{01}$$

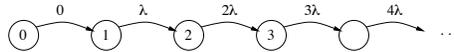
$$h_{11} = \frac{1}{\mu_2} + h_{10}$$

For If $\lambda = \mu_1 = \mu_2 = 1.0$ this yields $\begin{pmatrix} h_{00} \\ h_{01} \\ h_{10} \\ h_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 5 \end{pmatrix}$. Thus,

$h_{11} = 5$ is the required answer.

1.7. Simple population growth models

(a)



(b) $\pi_0(t) \equiv 0$. The Kolmogorov forward equations are given, for $k \geq 1$, by

$$\frac{\partial \pi_k}{\partial t} = (k-1)\lambda \pi_{k-1}(t) - k\lambda \pi_k(t).$$

Multiplying each side by λ^k and summing yields

$$\begin{aligned} \frac{\partial P(z, t)}{\partial t} &= \sum_{k=1}^{\infty} \frac{\partial \pi_k(t)}{\partial t} z^k = \lambda \left[z^2 \sum_{k=2}^{\infty} \pi_{k-1}(t) (k-1) z^{k-1} - z \sum_{k=1}^{\infty} \pi_k(t) k z^{k-1} \right] \\ &= \lambda(z^2 - z) \frac{\partial P(z, t)}{\partial z} \end{aligned}$$

That is, $P(z, t)$ is a solution of the partial differential equation:

$$\frac{\partial P(z, t)}{\partial t} = \lambda(z^2 - z) \frac{\partial P(z, t)}{\partial z}$$

with the initial condition $P(z, 0) = z$. It is easy to check that the expression given in the problem statement is a solution. Although it wasn't requested in the problem, here is a method to find the solution. The above PDE is a first order linear hyperbolic (i.e. wave type) equation which is well-posed and is readily solved by the method of characteristics. The idea is to rewrite the equation as $[\frac{\partial}{\partial t} - \lambda(z^2 - z) \frac{\partial}{\partial z}]P(z, t) = 0$, which means that $P(z, t)$ has directional derivative zero in the direction $(-\lambda(z - z^2), 1)$ in the (z, t) plane.

(c) Write $\bar{N}_t = E[N(t)]$. We could use the expression for $P(z, t)$ given and compute $\bar{N}_t =$

$\frac{\partial P(z,t)}{\partial z}|_{z=0}$. Alternatively, divide each side of the PDE by $z - 1$. Then since $\frac{\partial(1)}{\partial t} = 0$, we get $\frac{\partial}{\partial t} \left[\frac{P(z,t)-1}{z-1} \right] = \lambda z \frac{\partial P(z,t)}{\partial z}$. Letting $z \rightarrow 1$ then yields the differential equation $\frac{d\bar{N}_t}{dt} = \lambda \bar{N}_t$, so that $\bar{N}_t = e^{\lambda t}$.

(d) By part (b), $P(z,t) = \frac{ze^{-\lambda t}}{1-z(1-e^{-\lambda t})} = \sum_{k=1}^{\infty} e^{-\lambda t} (1-e^{-\lambda t})^{k-1} z^k$ so that $\pi_k(t) = e^{-\lambda t} (1-e^{-\lambda t})^{k-1}$, for $k \geq 1$. That is, the population size at time t has the geometric distribution with parameter $e^{-\lambda t}$.

(e) For the deterministic model, $n_t = 2^{\lfloor \lambda t \rfloor} \approx 2^{\lambda t} = e^{(\ln 2)\lambda t}$. Thus, the growth rate for the deterministic model is roughly exponential with exponent $(\ln 2)\lambda \approx (0.693)\lambda$, which is considerably smaller than the exponent λ for the random model.

1.8. Equilibrium distribution of the jump chain

The choice of B insures that $(-Bq_{ii}\pi_i)_{i \in \mathcal{S}}$ is a probability distribution. Since $p_{ij}^J = \begin{cases} 0 & \text{if } i = j \\ \frac{q_{ij}}{-q_{ii}} & \text{if } i \neq j \end{cases}$, we have for fixed j ,

$$\begin{aligned} \sum_{i \in \mathcal{S}} (-Bq_{ii}\pi_i)p_{ij}^J &= \sum_{i:i \neq j} -Bq_{ii}\pi_i \left(\frac{q_{ij}}{-q_{ii}} \right) \\ &= B \sum_{i:i \neq j} \pi_i q_{ij} = -B\pi_j q_{jj} \end{aligned}$$

Thus, $-B\pi_i q_{ii}$ is the equilibrium vector for P^J . In other words, $-B\pi_i q_{ii} = \pi_i^J$.

Another justification of the same result goes as follows. For the original Markov process, $-q_{ii}^{-1}$ is the mean holding time for each visit to state i . Since π_i^J represents the proportion of jumps of the process which land in state i , we must have $\pi_i \propto \pi_i^J (-q_{ii})^{-1}$, or equivalently, $\pi_i^J \propto (-q_{ii})\pi_i$.

1.9. A simple Poisson process calculation

Suppose $0 < s < t$ and $0 \leq i \leq k$.

$$\begin{aligned} P[N(s) = i | N(t) = k] &= \frac{P[N(s) = i, N(t) = k]}{P[N(t) = k]} \\ &= \left(\frac{e^{-\lambda s} (\lambda s)^i}{i!} \right) \left(\frac{e^{-\lambda(t-s)} (\lambda(t-s))^{k-i}}{(k-i)!} \right) \left(\frac{e^{-\lambda t} (\lambda t)^k}{k!} \right)^{-1} \\ &= \binom{k}{i} \left(\frac{s}{t} \right)^i \left(\frac{t-s}{t} \right)^{k-i} \end{aligned}$$

That is, given $N(t) = k$, the conditional distribution of $N(s)$ is binomial. This could have been deduced with no calculation, using the fact that given $N(t) = k$, the locations of the k points are uniformly and independently distributed on the interval $[0, t]$.

1.10. An alternating renewal process

(a) Sampled periods between light changes L are deterministic, length one, so the forward residual lifetime γ is uniformly distributed over $[0, 1]$. Thus, $P[\text{light changes}] = P[\gamma \leq 0.5] = 0.5$.

(b) By symmetry, γ is independent of what color is spotted first. If green is spotted first,

$$W = \begin{cases} 0 & \text{if } \gamma > 0.5 \\ \gamma + 0.5 & \text{if } \gamma < 0.5 \end{cases} \quad \text{so that } E[W | \text{green spotted}] = \int_0^{0.5} (u + 0.5) du = \frac{3}{8}. \quad \text{Similarly,}$$

$E[W|\text{red spotted}] = \int_{0.5}^1 (u - 0.5) du = \frac{1}{8}$. Thus, $E[W] = \frac{3}{8} \frac{1}{2} + \frac{1}{8} \frac{1}{2} = \frac{1}{4}$. This makes sense, because the light is red with probability one half when the light is reached, and given it is red when reached, the average wait is 0.5.

(c) $P[\text{light changes}] = P[\gamma \leq 0.5] = \int_0^{0.5} \frac{1 - F_X(y)}{m_1} dy = \int_0^{0.5} \frac{1 - \frac{y}{1}}{1} dy = \frac{7}{16}$.

(d) 2 blocks/minute (The point here is to express the answer in units given. If the length of a block is specified, the speed could be given in meters/second or miles per hour, for example.)

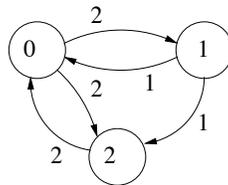
1.11. A simple question of periods

(a) The period of state 4 is $GCD\{4, 6, 8, 10, 12, 14, \dots\} = GCD\{4, 6\} = 2$.

(b) The process is irreducible, so all states have the same period. So state 6 must also have period 2.

1.12. A mean hitting time problem

(a)



$\pi Q = 0$ implies $\pi = (\frac{2}{7}, \frac{2}{7}, \frac{3}{7})$.

(b) Clearly $a_1 = 0$. Condition on the first step. The initial holding time in state i has mean $-\frac{1}{q_{ii}}$ and

the next state is j with probability $p_{ij}^J = \frac{-q_{ij}}{q_{ii}}$. Thus $\begin{pmatrix} a_0 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{q_{00}} \\ -\frac{1}{q_{22}} \end{pmatrix} + \begin{pmatrix} 0 & p_{02}^J \\ p_{20}^J & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_2 \end{pmatrix}$.

Solving yields $\begin{pmatrix} a_0 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$.

(c) Clearly $\alpha_2(t) = 0$ for all t .

$$\alpha_0(t+h) = \alpha_0(t)(1 + q_{00}h) + \alpha_1(t)q_{10}h + o(h)$$

$$\alpha_1(t+h) = \alpha_0(t)q_{01}h + \alpha_1(t)(1 + q_{11}h) + o(h)$$

Subtract $\alpha_i(t)$ from each side and let $h \rightarrow 0$ to yield $(\frac{\partial \alpha_0}{\partial t}, \frac{\partial \alpha_1}{\partial t}) = (\alpha_0, \alpha_1) \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}$ with the initial condition $(\alpha_0(0), \alpha_1(0)) = (1, 0)$. (Note: the matrix involved here is the Q matrix with the row and column for state 2 removed.)

(d) Similarly,

$$\beta_0(t-h) = (1 + q_{00}h)\beta_0(t) + q_{01}h\beta_1(t) + o(h)$$

$$\beta_1(t-h) = q_{10}h\beta_0(t) + (1 + q_{11}h)\beta_1(t) + o(h)$$

Subtract $\beta_i(t)$'s, divide by h and let $h \rightarrow 0$ to get: $\begin{pmatrix} -\frac{\partial \beta_0}{\partial t} \\ -\frac{\partial \beta_1}{\partial t} \end{pmatrix} = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ with

$$\begin{pmatrix} \beta_0(t_f) \\ \beta_1(t_f) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

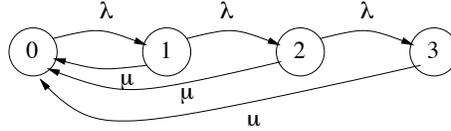
1.13. A birth-death process with periodic rates

(a) Let $\theta = \frac{\lambda_a \lambda_b}{\mu_a \mu_b}$. Then $S_1 = \sum_{n=0}^{\infty} \theta^n (1 + \frac{\lambda_a}{\mu_a})$, which is finite if and only if $\theta < 1$. Thus, the

process is positive recurrent if and only if $\theta < 1$. (In case $\theta < 1$, $\pi_{2n} = \theta^n/S_1$ and $\pi_{2n+1} = \theta^n \frac{\lambda_a}{S_1 \mu_a}$.)
 (b) $\lambda_a = \lambda_b$. In general, $r_k = \pi_k \lambda_k / \bar{\lambda}$, so that $r \equiv \pi$ if and only if the arrival rates are all equal (corresponding to Poisson arrivals, and then PASTA holds.)

1.14. Markov model for a link with resets

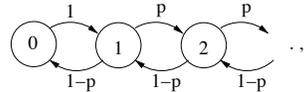
(a) Let $\mathcal{S} = \{0, 1, 2, 3\}$, where the state is the number of packets passed since the last reset.



(b) By the PASTA property, the dropping probability is π_3 . We can find the equilibrium distribution π by solving the equation $\pi Q = 0$. The balance equation for state 0 is $\lambda\pi_0 = \mu(1 - \pi_0)$ so that $\pi_0 = \frac{\mu}{\lambda + \mu}$. The balance equation for state $i \in \{1, 2\}$ is $\lambda\pi_{i-1} = (\lambda + \mu)\pi_i$, so that $\pi_1 = \pi_0(\frac{\lambda}{\lambda + \mu})$ and $\pi_2 = \pi_0(\frac{\lambda}{\lambda + \mu})^2$. Finally, $\lambda\pi_2 = \mu\pi_3$ so that $\pi_3 = \pi_0(\frac{\lambda}{\lambda + \mu})^2 \frac{\lambda}{\mu} = \frac{\lambda^3}{(\lambda + \mu)^3}$. The dropping probability is $\pi_3 = \frac{\lambda^3}{(\lambda + \mu)^3}$. (This formula for π_3 can be deduced with virtually no calculation from the properties of merged Poisson processes. Fix a time t . Each event is a packet arrival with probability $\frac{\lambda}{\lambda + \mu}$ and is a reset otherwise. The types of different events are independent. Finally, $\pi_3(t)$ is the probability that the last three events before time t were arrivals. The formula follows.)

1.15. An unusual birth-death process

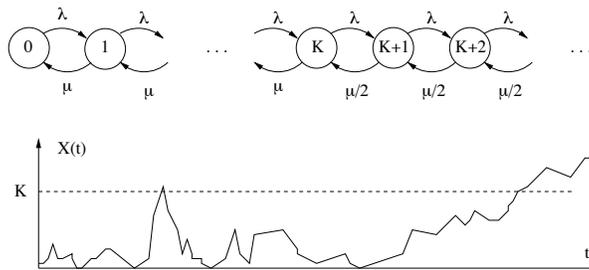
- (a) $S_2 = \sum_{n=0}^{\infty} \frac{\mu_1 \dots \mu_n}{\lambda_1 \dots \lambda_n} = \sum_{n=0}^{\infty} (\frac{1-p}{p})^n < +\infty$ (because $p > 0.5$) so X is transient.
- (b) $aQ = 0$ is equivalent to $\lambda_k a_k = \mu_k a_{k+1}$ for $k \geq 0$, which is easily checked.
- (c) $p_{01}^J = 1$ and, for $i \geq 1$, $p_{ii-1}^J = 1 - p$ and $p_{ii+1}^J = p$. All other transition probabilities are zero.



(d) S_2 is the same as in (a), so X^J is transient (as is also obvious from part (c).)
 Note: X reaches the graveyard state Δ in finite time. Otherwise, the fact that $Qa = 0$ for a probability distribution a would imply that all states are positive recurrent.

1.16. A queue with decreasing service rate

(a)



(b) $S_2 = \sum_{k=0}^{\infty} (\frac{\mu}{2\lambda})^k 2^{k \wedge K}$, where $k \wedge K = \min\{k, K\}$. Thus, if $\lambda < \frac{\mu}{2}$ then $S_2 < +\infty$ and the process is recurrent. $S_1 = \sum_{k=0}^{\infty} (\frac{2\lambda}{\mu})^k 2^{-k \wedge K}$, so if $\lambda < \frac{\mu}{2}$ then $S_1 < +\infty$ and the process is positive

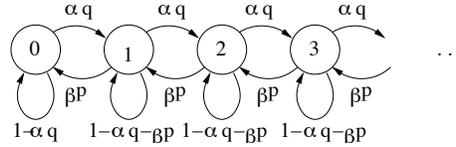
recurrent. In this case, $\pi_k = (\frac{2\lambda}{\mu})2^{-k \wedge K} \pi_0$, where

$$\pi_0 = \frac{1}{S_1} = \left[\frac{1 - (\lambda/\mu)^K}{1 - (\lambda/\mu)} + \frac{(\lambda/\mu)^K}{1 - (2\lambda/\mu)} \right]^{-1}.$$

(c) If $\lambda = \frac{2\mu}{3}$, the queue appears to be stable until it fluctuates above K . Eventually the queue-length will grow to infinity at rate $\lambda - \frac{\mu}{2} = \frac{\mu}{6}$. See figure above.

1.17. Limit of a discrete time queueing system

(q) The transition rate diagram for the number in the system is shown:

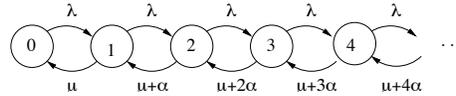


$a_k = P[\text{no arrivals for } k-1 \text{ slots, then arrival}] = (1 - \alpha q)^{k-1} \alpha q$ for $k \geq 1$. Similarly, $b_k = (1 - \beta q)^{k-1} \beta q$ for $k \geq 1$. Thus, an interarrival time is q times a geometric random variable with parameter αq , and the mean interarrival time is $\frac{1}{\alpha}$. and a service time is q times a geometric random variable with parameter βq , and the mean service time is $\frac{1}{\beta}$.

(b) $\beta q p_{k+1} = \alpha q p_k$ for $k \geq 0$, implies that $p_k = (1 - \rho) \rho^k$, where $\rho = \frac{\alpha}{\beta}$. This distribution is the same for all q , so it trivially converges as $q \rightarrow 0$ to the same value: $p_k \rightarrow (1 - \rho) \rho^k$. The interarrival times and service times become exponentially distributed in the limit. The limit system is a continuous time M/M/1 system.

1.18. An M/M/1 queue with impatient customers

(a)



(b) The process is positive recurrent for all λ, μ if $\alpha > 0$, and $p_k = \frac{c\lambda^k}{\mu(\mu+\alpha)\dots(\mu+(k-1)\alpha)}$ where c is chosen so that the p_k 's sum to one.

(c) If $\alpha = \mu$, $p_k = \frac{c\lambda^k}{k!\mu^k} = \frac{c\rho^k}{k!}$. Therefore, $(p_k : k \geq 0)$ is the Poisson distribution with mean ρ . Furthermore, p_D is the mean departure rate by defecting customers, divided by the mean arrival rate λ . Thus,

$$p_D = \frac{1}{\lambda} \sum_{k=1}^{\infty} p_k (k-1) \alpha = \frac{\rho - 1 + e^{-\rho}}{\rho} \rightarrow \begin{cases} 1 & \text{as } \rho \rightarrow \infty \\ 0 & \text{as } \rho \rightarrow 0 \end{cases}$$

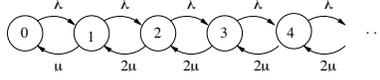
where l'Hôpital's rule can be used to find the limit as $\rho \rightarrow 0$.

1.19. Statistical multiplexing

(a) Use an M/M/1 model, with $\lambda = 40$ packets/second, and $\mu = 50$ packets per second. Then $T = \frac{1}{\lambda - \mu} = 0.1$ seconds.

(b) Model each link as an M/M/1 queue, with $\lambda = 20$ packets/second, and $\mu = 25$ packets per second. Then $T = \frac{1}{\lambda - \mu} = 0.2$ seconds.

(c) Model as an M/M/2 queue with $\lambda = 40$ packets/second and $\mu = 25$ packets/second.



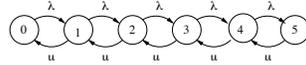
Let $\rho = \frac{\lambda}{2\mu}$. Then $\pi_0 = \frac{0.5}{0.5 + \rho + \rho^2 + \dots}$ and $\pi_k = 2\pi_0\rho^k$ for $k \geq 1$. Thus

$$T = \frac{\bar{N}}{\lambda} = \frac{\sum_{k=1}^{\infty} k\rho^k}{\lambda(0.5 + \rho + \rho^2 + \dots)} = \frac{\rho/(1-\rho)^2}{\lambda((\frac{1}{1-\rho}) - 0.5)} = 0.111 \text{ seconds}$$

We can view model (c) as a variation of (b) with statistical multiplexing at the packet level, and (a) as a variation with multiplexing at the subpacket level. Here (c) is a substantial improvement over (b), and (a) is a little better than (c).

1.20. A queue with blocking

(a)



$$\pi_k = \frac{\rho^k}{1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5} = \frac{\rho^k(1-\rho)}{1-\rho^6} \text{ for } 0 \leq k \leq 5.$$

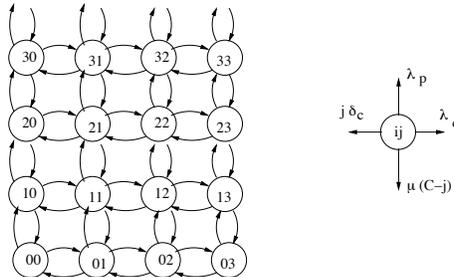
(b) $p_B = \pi_5$ by the PASTA property.

(c) $\bar{W} = \bar{N}_W / (\lambda(1 - p_B))$ where $\bar{N}_W = \sum_{k=0}^5 (k-1)\pi_k$. Alternatively, $\bar{W} = \bar{N} / (\lambda(1 - p_B)) - \frac{1}{\mu}$ (i.e. \bar{W} is equal to the mean time in system minus the mean time in service)

(d) $\pi_0 = \frac{1}{\lambda(\text{mean cycle time for visits to state zero})} = \frac{1}{\lambda(1/\lambda + \text{mean busy period duration})}$ Therefore, the mean busy period duration is given by $\frac{1}{\lambda}[\frac{1}{\pi_0} - 1] = \frac{\rho - \rho^6}{\lambda(1-\rho)} = \frac{\rho - \rho^5}{\mu(1-\rho)}$

1.21. Multiplexing circuit and packet data streams

(a) A portion of the rate diagram for $C = 3$ is pictured.



(b) Yes, the phase structure is apparent.

$$(c) \lambda_p \leq \mu(C - \bar{n}_C) = \frac{\mu \sum_{j=0}^C (C-j) (\frac{\lambda_c}{\delta_C})^j / j!}{\sum_{j=0}^C (\frac{\lambda_c}{\delta_C})^j / j!}$$

1.22. Three queues and an autonomously traveling server

(a) The equilibrium distribution for the server is $\pi = (\pi_1, \pi_2, \pi_3) = (\frac{1}{\alpha Z}, \frac{1}{\beta Z}, \frac{1}{\gamma Z})$, where $Z = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$, and the service capacity of station i is thus π_i , for $1 \leq i \leq 3$.

(b) The equilibrium distributions for the individual stations can be found one at a time, by the method of phases (even though the stations are not independent). For a fixed station i , the state space would be $\{(l, \theta) : l \geq 0, 1 \leq \theta \leq 3\}$, where l is the number of customers at station i and θ is

Kingman's moment bound, disussed later in the notes: $\bar{X} \leq \frac{\text{Var}(B)}{-2\bar{B}}$.)

2.3. Routing with two arrival streams

We begin by describing the system by stochastic evolution equations. For $i \in \{0, 1\}$, let $B_i(t)$ be a Bernoulli(a_i) random variable, and let $U_i(t)$ be a Bernoulli(u_i) random variable, and for $j \in \{1, 2, 3\}$ let $D_j(t)$ be a Bernoulli(d_j) random variable. Suppose all these Bernoulli random variables are independent. Then the state process can be described by

$$X_i(t+1) = X_i(t) + A_i(t) - D_i(t) + L_i(t) \quad \text{for } i \in \{1, 2, 3\} \quad (10.4)$$

where

$$\begin{aligned} A_1(t) &= U_1(t)B_1(t) \\ A_2(t) &= (1 - U_1(t))B_1(t) + U_2(t)B_2(t) \\ A_3(t) &= (1 - U_2(t))B_2(t) \end{aligned}$$

and $L_i(t) = (-(X_i(t) + A_i(t) - D_i(t)))_+$.

Assume that the parameters satisfy the following three conditions:

$$a_1 + a_2 < d_1 + d_2 + d_3 \quad (10.5)$$

$$a_1 < d_1 + d_2 \quad (10.6)$$

$$a_2 < d_2 + d_3 \quad (10.7)$$

These conditions are necessary for any routing strategy to yield positive recurrence. Condition (10.5) states that the total arrival rate to the network is less than the sum of service rates in the network. Condition (10.6) states that the total arrival rate to servers 1 and 2 is less than the sum of the service rates of those servers. Condition (10.7) has a similar interpretation. We will show that conditions (10.5)-(10.7) are also sufficient for positive recurrence for random routing for a particular choice of u . To that end we consider the Lyapunov-Foster criteria.

Letting $V = (x_1^2 + x_2^2 + x_3^2)$ and arguing as in Example 1a, and using the fact $(A_i(t) - D_i(t))^2 \leq 1$ for $i = 1, 3$ and $(A_2(t) - D_2(t))^2 \leq 4$ yields (writing \bar{u}_i for $1 - u_i$):

$$\begin{aligned} PV(x) - V(x) &\leq \left(\sum_{i=1}^3 x_i E[A_i(t) - D_i(t) | X(t) = x] \right) + 3 \\ &= x_1(a_1 u_1 - d_1) + x_2(a_1 \bar{u}_1 + a_2 u_2 - d_2) + x_3(a_2 \bar{u}_2 - d_3) + 3 \end{aligned} \quad (10.8)$$

$$\leq -(x_1 + x_2 + x_3) \min\{d_1 - a_1 u_1, d_2 - a_1 \bar{u}_1 - a_2 u_2, d_3 - a_2 \bar{u}_2\} + 3 \quad (10.9)$$

To obtain the best bound possible, we select $u = (u_1, u_2)$ to maximize the min term in (10.9). For any u , $\min\{d_1 - a_1 u_1, d_2 - a_1 \bar{u}_1 - a_2 u_2, d_3 - a_2 \bar{u}_2\}$, is the minimum of three numbers, which is less than or equal to the average of the three numbers, is less than or equal to the average of any two of the three numbers, and is less than or equal to any one of the three numbers. Therefore, for any choice of $u \in [0, 1]^2$,

$$\min\{d_1 - a_1 u_1, d_2 - a_1 \bar{u}_1 - a_2 u_2, d_3 - a_2 \bar{u}_2\} \leq \epsilon \quad (10.10)$$

where

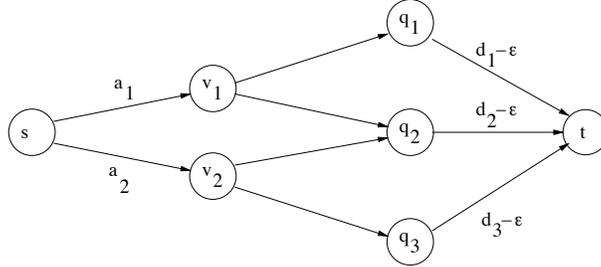
$$\epsilon = \min\left\{ \frac{d_1 + d_2 + d_3 - a_1 - a_2}{3}, \frac{d_1 + d_2 - a_1}{2}, \frac{d_2 + d_3 - a_2}{2}, d_1, d_2, d_3 \right\}$$

Note that $\epsilon > 0$ under the necessary conditions (10.5)-(10.7). Taking $u_1^* \in [0, 1]$ as large as possible subject to $d_1 - a_1 u_1^* \geq \epsilon$, and taking $\bar{u}_2^* \in [0, 1]$ as large as possible subject to $d_3 - a_2 \bar{u}_2^* \geq \epsilon$, yields the choice

$$u_1^* = \frac{d_1 - \epsilon}{a_1} \wedge 1 \quad \bar{u}_2^* = \frac{d_3 - \epsilon}{a_2} \wedge 1$$

It is not hard to check that $d_2 - a_1 \bar{u}_1^* - a_2 \bar{u}_2^* \geq \epsilon$, so that equality holds in (10.10) for $u = u^*$.

This paragraph shows that the above choice of ϵ and u^* can be better understood by applying the max-flow min-cut theorem to the flow graph shown.



In addition to the source node s and sink node t , there are two columns of nodes in the graph. Nodes v_1 and v_2 correspond to the two arrival streams, and nodes q_1, q_2 and q_3 correspond to the three queues. There are three stages of links in the graph. The capacity of a link (s, v_i) in the first stage is a_i , the capacities of the links in the second graph are very large, and the capacity of a link (q_j, t) in the third stage is $d_j - \epsilon$. The choice of ϵ above is the largest possible so that (1) all links have nonnegative capacity, and (2) the capacity of any $s - t$ cut is greater than or equal to $a_1 + a_2$. Thus, the min flow max cut theorem insures that there exists an $s - t$ flow with value $a_1 + a_2$. Then, u_i^* can be taken to be the fraction of the flow into v_i that continues to q_i , for $i = 1$ and $i = 2$.

Under (10.5)-(10.7) and using u^* , we have $PV(x) - V(x) \leq -\epsilon(x_1 + x_2 + x_3) + 3$. Thus, the system is positive recurrent, and $\bar{X}_1 + \bar{X}_2 + \bar{X}_3 \leq \frac{3}{\epsilon}$.¹

(b) Suppose each arrival is routed to the shorter of the two queues it can go to. For simplicity, let such decisions be based on the queue lengths at the beginning of the slot. As far as simple bounds are concerned, there is no need, for example, to take into account the arrival in the top branch when making the decision on the bottom branch, although presumably this could improve performance. Also, it might make sense to break ties in favor of queues 1 and 3 over queue 2, but that is not very important either. We can denote this route-to-shorter policy by thinking of u in part (a) as a function of the state, $u = \phi^{RS}(x)$. Note that for given x , this choice of u minimizes the right hand side of (10.8). In particular, under the route-to-shorter policy, the right hand side of (10.8) is at least as small as its value for the fixed control u^* . Therefore, writing P^{RS} for the one-step transition matrix for the route to shorter policy, $P^{RS}V(x) - V(x) \leq -\epsilon(x_1 + x_2 + x_3) + 3$, and under the necessary conditions (10.5)-(10.7), the system is positive recurrent, and $\bar{X}_1 + \bar{X}_2 + \bar{X}_3 \leq \frac{3}{\epsilon}$.

2.4. An inadequacy of a linear potential function

Suppose x is on the positive x_2 axis (i.e. $x_1 = 0$ and $x_2 > 0$). Then, given $X(t) = x$, during the slot, queue 1 will increase to 1 with probability $a(1 - d_1) = 0.42$, and otherwise stay at zero.

¹By using the fact that $\bar{X}_i = 0$ if the arrival rate to queue i is zero, it can be shown that for $u = u^*$, $\bar{X}_1 + \bar{X}_2 + \bar{X}_3 \leq \frac{3}{\epsilon'}$, where $\epsilon' = \min\{\frac{d_1 + d_2 + d_3 - a_1 - a_2}{3}, \frac{d_1 + d_2 - a_1}{2}, \frac{d_2 + d_3 - a_2}{2}\}$.

Queue 2 will decrease by one with probability 0.4, and otherwise stay the same. Thus, the drift of V , $E[V(X(t+1) - V(x)|X(t) = x)]$ is equal to 0.02. Therefore, the drift is strictly positive for infinitely many states, whereas the Foster-Lyapunov condition requires that the drift be negative off of a finite set C . So, the linear choice for V does not work for this example.

2.5. Allocation of service

Let $\epsilon > 0$. The largest we can make $\mu_1(u)$ is m_1 and the largest we can make $\mu_3(u)$ is m_2 . Thus, we can select u so that

$$\lambda_1 + \epsilon \leq \mu_1(u) \quad \text{and} \quad \lambda_3 + \epsilon \leq \mu_3(u) \quad (10.11)$$

if and only if $m_1 \geq \lambda_1 + \epsilon$ and $m_2 \geq \lambda_3 + \epsilon$. If we select u so that equality holds in (10.11) then the value of $\mu_2(u)$ is maximized subject to (10.11), and it is given by

$$\mu_2(u) = m_1 + m_2 - \mu_1(u) - \mu_3(u) = m_1 + m_2 - \lambda_1 + \lambda_3 - 2\epsilon$$

Therefore, in order that $\lambda_2 + \epsilon \leq \mu_2(u)$ hold in addition to (10.11), it is necessary and sufficient that $\epsilon \leq \frac{\lambda_1 + \lambda_2 + \lambda_3 - \mu_1 - \mu_2}{3}$ and $m_1 \geq \lambda_1 + \epsilon$ and $m_2 \geq \lambda_2 + \epsilon$. Thus, the value of ϵ given in the example is indeed the largest possible.

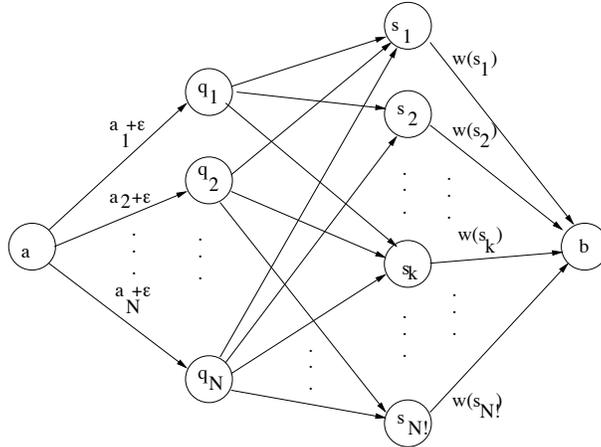
2.6. Opportunistic scheduling (Tassiulas and Ephremides [40])

(a) The left hand side of (2.27) is the arrival rate to the set of queues in s , and the righthand side is the probability that some queue in s is eligible for service in a given time slot. The condition is necessary for the stability of the set of queues in s .

(b) Fix $\epsilon > 0$ so that for all $s \in E$ with $s \neq \emptyset$,

$$\sum_{i \in s} (a_i + \epsilon) \leq \sum_{B: B \cap s \neq \emptyset} w(B)$$

Consider the flow graph shown.



In addition to the source node a and sink node b , there are two columns of nodes in the graph. The first column of nodes corresponds to the N queues, and the second column of nodes corresponds to the 2^N subsets of E . There are three stages of links in the graph. The capacity of a link (a, q_i) in the first stage is $a_i + \epsilon$, there is a link (q_i, s_j) in the second stage if and only if $q_i \in s_j$, and each

such link has capacity greater than the sum of the capacities of all the links in the first stage, and the weight of a link (s_k, t) in the third stage is $w(s_k)$.

We claim that the minimum of the capacities of all $a - b$ cuts is $v^* = \sum_{i=1}^N (a_i + \epsilon)$. Here is a proof of the claim. The $a - b$ cut $(\{a\} : V - \{a\})$ (here V is the set of nodes in the flow network) has capacity v^* , so to prove the claim, it suffices to show that any other $a - b$ cut has capacity greater than or equal to v^* . Fix any $a - b$ cut $(A : B)$. Let $\tilde{A} = A \cap \{q_1, \dots, q_N\}$, or in words, \tilde{A} is the set of nodes in the first column of the graph (i.e. set of queues) that are in A . If $q_i \in \tilde{A}$ and $s_j \in B$ such that (q_i, s_j) is a link in the flow graph, then the capacity of $(A : B)$ is greater than or equal to the capacity of link (q_i, s_j) , which is greater than v^* , so the required inequality is proved in that case. Thus, we can suppose that A contains all the nodes s_j in the second column such that $s_j \cap \tilde{A} \neq \emptyset$. Therefore,

$$\begin{aligned} C(A : B) &\geq \sum_{i \in \{q_1, \dots, q_N\} - \tilde{A}} (a_i + \epsilon) + \sum_{s \in E : s \cap \tilde{A} \neq \emptyset} w(s) \\ &\geq \sum_{i \in \{q_1, \dots, q_N\} - \tilde{A}} (a_i + \epsilon) + \sum_{i \in \tilde{A}} (a_i + \epsilon) = v^*, \end{aligned} \quad (10.12)$$

where the inequality in (10.12) follows from the choice of ϵ . The claim is proved.

Therefore there is an $a - b$ flow f which saturates all the links of the first stage of the flow graph. Let $u(i, s) = f(q_i, s)/f(s, b)$ for all i, s such that $f(s, b) > 0$. That is, $u(i, s)$ is the fraction of flow on link (s, b) which comes from link (q_i, s) . For those s such that $f(s, b) = 0$, define $u(i, s)$ in some arbitrary way, respecting the requirements $u(i, s) \geq 0$, $u(i, s) = 0$ if $i \notin s$, and $\sum_{i \in E} u(i, s) = I_{\{s \neq \emptyset\}}$. Then $a_i + \epsilon = f(a, q_i) = \sum_s f(q_i, s) = \sum_s f(s, b)u(i, s) \leq \sum_s w(s)u(i, s) = \mu_i(u)$, as required.

(c) Let $V(x) = \frac{1}{2} \sum_{i \in E} x_i^2$. Let $\delta(t)$ denote the identity of the queue given a potential service at time t , with $\delta(t) = 0$ if no queue is given potential service. Then $P[\delta(t) = i | S(t) = s] = u(i, s)$. The dynamics of queue i are given by $X_i(t+1) = X_i(t) + A_i(t) - R_i(\delta(t)) + L_i(t)$, where $R_i(\delta) = I_{\{\delta=i\}}$. Since $\sum_{i \in E} (A_i(t) - R_i(\delta_i(t)))^2 \leq \sum_{i \in E} (A_i(t))^2 + (R_i(\delta_i(t)))^2 \leq N + \sum_{i \in E} A_i(t)^2$ we have

$$PV(x) - V(x) \leq \left(\sum_{i \in E} x_i (a_i - \mu_i(u)) \right) + K \quad (10.13)$$

$$\leq -\epsilon \left(\sum_{i \in E} x_i \right) + K \quad (10.14)$$

where $K = \frac{N}{2} + \sum_{i=1}^N K_i$. Thus, under the necessary stability conditions we have that under the vector of scheduling probabilities u , the system is positive recurrent, and

$$\sum_{i \in E} \bar{X}_i \leq \frac{K}{\epsilon} \quad (10.15)$$

(d) If u could be selected as a function of the state, x , then the right hand side of (10.13) would be minimized by taking $u(i, s) = 1$ if i is the smallest index in s such that $x_i = \max_{j \in s} x_j$. This suggests using the *longest connected first* (LCF) policy, in which the longest connected queue is served in each time slot. If P^{LCF} denotes the one-step transition probability matrix for the LCF policy, then (10.13) holds for any u , if P is replaced by P^{LCF} . Therefore, under the necessary condition and ϵ as in part (b), (10.14) also holds with P replaced by P^{LCF} , and (10.15) holds for

the LCF policy.

2.7. Routing to two queues – continuous time model

Suppose $\lambda < \mu_1 + \mu_2$, which is a necessary condition for positive recurrence.

(a) Under this condition, we can find u so that $\mu_1 > \lambda u$ and $\mu_2 > \lambda \bar{u}$. If each customer is independently routed to queue 1 with probability u , and if $V(x) = (x_1^2 + x_2^2)/2$, (2.20) becomes

$$2QV(x) = ((x_1 + 1)^2 - x_1^2) \lambda u + ((x_2 + 1)^2 - x_2^2) \lambda \bar{u} + ((x_1 - 1)_+^2 - x_1^2) \mu_1 + ((x_2 - 1)_+^2 - x_2^2) \mu_2.$$

Since $(x_1 - 1)_+^2 \leq (x_1 - 1)^2$, it follows, with $\gamma = \lambda + \mu_1 + \mu_2$, that

$$QV(x) \leq -(x_1(\mu_1 - u\lambda) + x_2(\mu_2 - \bar{u}\lambda)) + \frac{\gamma}{2}. \quad (10.16)$$

Thus, the combined stability criteria and moment bound applies, yielding that the process is positive recurrent, and $\bar{X}_1(\mu_1 - \lambda) + \bar{X}_2(\mu_2 - \bar{u}\lambda) \leq \frac{\gamma}{2}$.

In analogy to Example 1a, we let

$$\begin{aligned} \epsilon &= \max_{0 \leq u \leq 1} \min\{\mu_1 - \lambda u, \mu_2 - \lambda \bar{u}\} \\ &= \min\left\{\mu_1, \mu_2, \frac{\mu_1 + \mu_2 - \lambda}{2}\right\} \end{aligned}$$

and the corresponding value u^* of u is given by

$$u^* = \begin{cases} 0 & \text{if } \mu_1 - \mu_2 < -\lambda \\ \frac{1}{2} + \frac{\mu_1 - \mu_2}{2\lambda} & \text{if } |\mu_1 - \mu_2| \leq \lambda \\ 1 & \text{if } \mu_1 - \mu_2 > \lambda \end{cases}$$

For the system with $u = u^*$, $\bar{X}_1 + \bar{X}_2 \leq \frac{\gamma}{2\epsilon}$. The remark at the end of Example 1a also carries over, yielding that for splitting probability $u = u^*$, $\bar{X}_1 + \bar{X}_2 \leq \frac{\gamma}{\mu_1 + \mu_2 - \lambda}$.

(b) Consider now the case that when a packet arrives, it is routed to the shorter queue. To be definite, in case of a tie, the packet is routed to queue 1. Let Q^{RS} denote the transition rate matrix in case the route to short queue policy is used. Note that for any u , $\lambda(x_1 u + x_2 \bar{u}) \geq \lambda(x_1 \wedge x_2)$. Thus, (10.16) continues to hold if Q is replaced by Q^{RS} . In particular, if $\lambda < \mu_1 + \mu_2$, then the process X under the route-to-shorter routing is positive recurrent, and $\bar{X}_1 + \bar{X}_2 \leq \frac{\gamma}{2\epsilon}$.

2.8. Stability of two queues with transfers

(a) System is positive recurrent for some u if and only if $\lambda_1 < \mu_1 + \nu$, $\lambda_2 < \mu_2$, and $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$.

(b)

$$\begin{aligned} QV(x) &= \sum_{y:y \neq x} q_{xy} (V(y) - V(x)) \\ &= \frac{\lambda_1}{2} [(x_1 + 1)^2 - x_1^2] + \frac{\lambda_2}{2} [(x_2 + 1)^2 - x_2^2] + \frac{\mu_1}{2} [(x_1 - 1)_+^2 - x_1^2] + \\ &\quad \frac{\mu_2}{2} [(x_2 - 1)_+^2 - x_2^2] + \frac{u\nu I_{\{x_1 \geq 1\}}}{2} [(x_1 - 1)^2 - x_1^2 + (x_2 + 1)^2 - x_2^2] \quad (10.17) \end{aligned}$$

(c) If the righthand side of (10.17) is changed by dropping the positive part symbols and dropping the factor $I_{\{x_1 \geq 1\}}$, then it is not increased, so that

$$\begin{aligned} QV(x) &\leq x_1(\lambda_1 - \mu_1 - u\nu) + x_2(\lambda_2 + u\nu - \mu_2) + K \\ &\leq -(x_1 + x_2) \min\{\mu_1 + u\nu - \lambda_1, \mu_2 - \lambda_2 - u\nu\} + K \quad (10.18) \end{aligned}$$

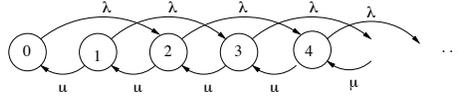
where $K = \frac{\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + 2\nu}{2}$. To get the best bound on $\overline{X_1} + \overline{X_2}$, we select u to maximize the min term in (10.18), or $u = u^*$, where u^* is the point in $[0, 1]$ nearest to $\frac{\mu_1 + \mu_2 - \lambda_1 - \lambda_2}{2\nu}$. For $u = u^*$, we find $QV(x) \leq -\epsilon(x_1 + x_2) + K$ where $\epsilon = \min\{\mu_1 + \nu - \lambda_1, \mu_2 - \lambda_2, \frac{\mu_1 + \mu_2 - \lambda_1 - \lambda_2}{2}\}$. Which of the three terms is smallest in the expression for ϵ corresponds to the three cases $u^* = 1$, $u^* = 0$, and $0 < u^* < 1$, respectively. It is easy to check that this same ϵ is the largest constant such that the stability conditions (with strict inequality relaxed to less than or equal) hold with (λ_1, λ_2) replaced by $(\lambda_1 + \epsilon, \lambda_2 + \epsilon)$.

2.9. Stability of a system with two queues and modulated server

By inspection, the equilibrium distribution of the server state is $(w(0), w(1), w(2)) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Thus, over the long run, the server can serve queue 1 at most $\frac{3}{4}$ of the time, it can serve queue 2 at most $\frac{3}{4}$ of the time, and, of course, it can offer at most one service in each slot. Thus, it is necessary that $a_1 < \frac{3}{4}$, $a_2 < \frac{3}{4}$, and $a_1 + a_2 < 1$. (It can be shown that these conditions are also sufficient for stability by using the Lyapunov function $V(x) = \frac{x_1^2 + x_2^2}{2}$ applied to the time-sampled process $Y(n) = X(nk)$, where k is a constant large enough so that the mean fraction of time the server is in each state over k time slots is close to the statistical average, no matter what the state of the server at the beginning of such interval.)

3.1. A queue with customers arriving in pairs

(a)



(b) For any function V on \mathbb{Z} , $QV(x) = \lambda(Q(x+2) - Q(x)) + \mu(Q((x-1)_+) - Q(x))$. In particular, if $V(x) = x$, then $QV(x) = -(\mu - 2\lambda) + \mu I_{\{x=0\}}$. Therefore, by the Foster-Lypunov stability criteria, if $\mu \geq 2\lambda$, then N is recurrent, and if $\mu > 2\lambda$, then N is positive recurrent.

(c) Take $V(x) = \frac{x^2}{2}$. Then,

$$QV(x) = \begin{cases} -x(\mu - 2\lambda) + \frac{4\lambda + \mu}{2} & \text{if } x \geq 1 \\ 2\lambda & \text{if } x = 0 \end{cases}$$

Thus $QV(x) \leq -x(\mu - 2\lambda) + \frac{4\lambda + \mu}{2}$ for all $x \in \mathbb{Z}$. Conclude by the combined Foster-Lyapunov criteria and moment bounds that if $\mu > 2\lambda$, then N is positive recurrent, and $\overline{N} \leq \frac{4\lambda + \mu}{2(\mu - 2\lambda)}$.

(d) Since the process is not explosive, it is positive recurrent if and only if there is a probability distribution p so that $pQ = 0$. The equation $pQ = 0$ means that the net probability flux into each state should be zero. Equivalently, $pQ = 0$ means that the next flux out of the set $\{0, 1, \dots, k\}$ is zero for each $k \geq 0$. Defining $p_{-1} = 0$ for convenience, the equation $pQ = 0$ is equivalent to the set of equations: $(p_{k-1} + p_k)\lambda = p_{k+1}\mu$ for $k \geq 0$. (This is a set of second order linear difference equations.) Multiply each side of the k^{th} equation by z^{k+1} , sum from $k = 0$ to ∞ to yield $(z^2\lambda + z\lambda)P(z) = \mu(P(z) - p_0)$, so that

$$P(z) = \frac{p_0}{1 - (\lambda z/\mu) - (\lambda z^2/\mu)}$$

The condition $P(1) = 1$ yields that $p_0 = 1 - 2\lambda/\mu$, which is valid only if $2\lambda/\mu < 1$. Under that

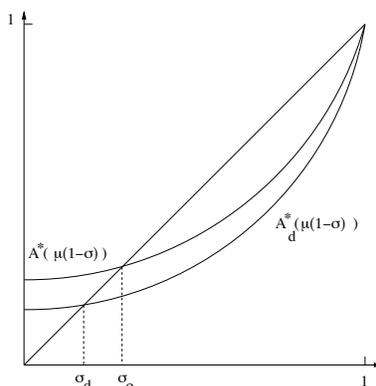
Then,

$$P_{loss \text{ for packets}} = P[\text{no tokens present}] = \frac{1}{1 + \frac{1}{\lambda} + \dots + \frac{1}{\lambda^B}} = \frac{\lambda^B}{\lambda^B + \lambda^{B-1} + \dots + 1}$$

3.3. Extremality of constant interarrival times for G/M/1 queues

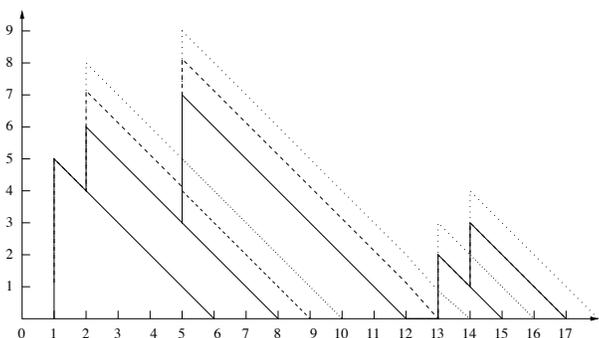
(a) $A_d^*(s) = \exp(-s/\lambda)$. Let t_1 have distribution function A . Then since $\exp(-sx)$ is convex in x , Jensen's inequality can be applied to yield $A^*(s) = E[\exp(-st_1)] \geq \exp(-sE[t_1]) = A_d^*(s)$.

(b) Let σ_o and σ_d be values of σ for the respective queues. Since $A_d^*(\mu(1-\sigma_o)) \leq A^*(\mu(1-\sigma_o)) = \sigma_o$ and A_d^* is convex, it follows (see the figure) that $\sigma_d \leq \sigma_o$. The assertions follow because the required means are increasing functions of σ .



3.4. Propagation of perturbations

(a) See the solid lines in the figure. The departure times are indicated by where the solid diagonal lines meet the axis, and occur at times 6,8,12,15, and 17. The waiting times in queue are 0, 4, 3, 0, 1.



(b) The departure times are indicated by where the dashed diagonal lines meet the axis, and occur at times 6,9,13,15, and 17. The waiting times in queue are 0, 4, 4, 0, 1.

(c) The departure times are indicated by where the dashed diagonal lines meet the axis, and occur at times 6,10, 14, 16, and 18. The waiting times in queue are 0, 4, 5, 1, 2.

(d) The waiting times of all following customers in the same busy period increase by the same amount. Busy periods can merge, and the waiting times of customers originally in a later busy period can also increase.

3.5. On priority M/GI/1 queues

(a) The server works on a type 1 customer whenever one is present, and otherwise works on a type 2 customer, if one is present. If the server is interrupted by a type 1 customer when it is working on a type 2 customer, the service of the type 2 customer is suspended until no more type 1 customers are in the system. Then service of the type 2 customer is resumed, with its remaining service time the same as when the service was interrupted.

(b) Let $\lambda = \lambda_1 + \lambda_2$, $\rho_i = \lambda_i \bar{X}_i$, and $\rho = \rho_1 + \rho_2$. Then $\bar{X} = \frac{\sum_i \lambda_i \bar{X}_i}{\lambda}$ and $\bar{X}^2 = \frac{\sum_i \lambda_i \bar{X}_i^2}{\lambda}$. Type 1 customers are not effected by type 2 customers, so the mean system time for type 1 customers is the same as if there were no type 2 customers:

$$T_1 = W_1 + \bar{X}_1 = \frac{\lambda_1 \bar{X}_1^2}{2(1 - \rho_1)} + \bar{X}_1$$

where $\rho_1 = \lambda_1 \bar{X}_1$. To find an expression for T_2 , consider a type 2 customer. Recall that the mean work in the system is given by $\frac{\lambda \bar{X}^2}{2(1 - \rho)}$. The PASTA property holds. Thus, T_2 is the mean of the sum of the work already in the system when the customer arrives, the service time of the customer itself, and the amount of preempting work that arrives while the type 2 customer is in the system:

$$T_2 = \frac{\lambda \bar{X}^2}{2(1 - \rho)} + \bar{X}_2 + \rho_1 T_2.$$

Solving for T_2 yields

$$T_2 = \left(\frac{1}{1 - \rho_1} \right) \left(\frac{\lambda \bar{X}^2}{2(1 - \rho)} + \bar{X}_2 \right)$$

3.6. Optimality of the μc rule

Let $J(\sigma) = \sum_i \lambda_i c_i W_i(\sigma) = \sum_i \mu_i c_i \rho_i W_i(\sigma)$ denote the cost for permutatation σ .

Claim: Let σ be a permutation and let $\hat{\sigma}$ denote another permutation obtained from σ by swapping σ_i and σ_{i+1} for some i . Then $J(\hat{\sigma}) < J(\sigma)$ if and only if $\mu_{\sigma_i} c_{\sigma_i} < \mu_{\sigma_{i+1}} c_{\sigma_{i+1}}$.

The claim is proved as follows. Since only customers of classes i and $i + 1$ are affected by the swap, $W_j(\hat{\sigma}) = W_j(\sigma)$ for $j \notin \{\sigma_i, \sigma_{i+1}\}$. On the other hand, since class σ_i (i.e. class $\hat{\sigma}_{i+1}$) customers have lower priority under $\hat{\sigma}$ than under σ , $W_{\sigma_i}(\hat{\sigma}) > W_{\sigma_i}(\sigma)$. Thus, $\delta > 0$, where

$$\delta = \rho_{\sigma_i} W_{\sigma_i}(\hat{\sigma}) - \rho_{\sigma_i} W_{\sigma_i}(\sigma).$$

Similarly, $W_{\sigma_{i+1}}(\hat{\sigma}) < W_{\sigma_{i+1}}(\sigma)$, and furthermore by the conservation of flow equations,

$$\rho_{\sigma_{i+1}} W_{\sigma_{i+1}}(\hat{\sigma}) - \rho_{\sigma_{i+1}} W_{\sigma_{i+1}}(\sigma) = -\delta.$$

Therefore, $J(\hat{\sigma}) - J(\sigma) = (\mu_{\sigma_i} c_{\sigma_i} - \mu_{\sigma_{i+1}} c_{\sigma_{i+1}}) \delta$. Since $\delta > 0$, the claim follows immediately.

By the claim, if σ does not satisfy the ordering condition $\mu_{\sigma_1} c_{\sigma_1} \geq \mu_{\sigma_2} c_{\sigma_2} \geq \dots \geq \mu_{\sigma_K} c_{\sigma_K}$, then σ is not optimal.

Conversely, if σ does satisfy the ordering condition and if σ^0 is an arbitrary ordering, there is a sequence of orderings $\sigma^0, \sigma^1, \dots, \sigma^p = \sigma$ so that $J(\sigma^0) \geq J(\sigma^1) \geq \dots \geq J(\sigma^p) = J(\sigma)$, so that σ is optimal.

The μc rule is intuitively plausible because μc is the rate that instantaneous cost is reduced when a rate μ server works on a customer with instantaneous holding cost c .

3.7. A queue with persistent customers

(a) The system is stable if and only if $\frac{\lambda D}{b} < 1$. See part (b) for justification.

(b) Under the PR service order, we can view the network as an ordinary M/GI/1 queue in which the service time variable X for a typical customer is D times a $\text{Geo}(b)$ random variable. The system is stable if $\rho < 1$, where $\rho = \lambda \bar{X} = \frac{\lambda D}{b}$. Under this condition, the mean time in the system is given by the Polleccek-Khinchine formula with $\rho = \frac{\lambda D}{b}$, $\bar{X} = \frac{D}{b}$, and $\overline{X^2} = \frac{D^2(2-b)}{b^2}$:

$$T = W + \bar{X} = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \bar{X} = \frac{(2-\lambda D)D}{2(b-\lambda D)}.$$

(c) The distribution of the number of customers in the system does not depend on the order of service (assuming no advance knowledge of how many times a given customer will return). So the mean number of customers in the system does not depend on the service order. So by Little's law, the mean time in the system does not depend on the order of service. However, the FIFO order at the queue would lead to a larger variance of total system time than the PR service order. (This is implied by the fact that if $a_1 < \dots < a_n$ and $d_1 < \dots < d_n$ and $a_i < d_i$, then $\sum (d_i - a_i)^2 < \sum (d_{\sigma_i} - a_i)^2$ for any permutation σ other than the identity permutation. Note that the PR order is not FIFO for the queue, but is FIFO for the system.)

3.8. A discrete time M/GI/1 queue

The mean service time is $7/2$, so the service rate is $2/7$. Let $\rho = 7p/2$, which is the arrival rate, p , divided by the service rate. Also, ρ is the probability the server is busy in a time slot. Arrivals see the system in equilibrium by the discrete time version of the PASTA property, which is true when numbers of arrivals in distinct slots are independent. Given the server is busy during the slot of an arrival, the residual service time γ of the customer in service is distributed over $\{0, 1, 2, 3, 4, 5\}$. The total service time of the customer in service has distribution $P[l = l] = \frac{l!}{m_1}$, for $1 \leq l \leq 6$, with mean $E[L] = \frac{m_2}{m_1} = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6m_1} = \frac{13}{3}$. Given L , γ is uniform on $\{0, 1, \dots, L-1\}$, so $E[\gamma] = \frac{E[L]-1}{2} = \frac{5}{3}$. Thus, $W = 3.5N + (\frac{7p}{2})(\frac{5}{3})$, where N is the mean number of customers seen by an arrival, equal to pW . Solving for W yields $W = \frac{35p}{6-21p}$ for $0 \leq p < \frac{2}{7}$. An embedded Markov process is given by the number of customers in the system just after the departure of a customer. To be specific, this number could include the new arrival, if any, occurring in the same slot that a service is completed.

Here is a second way to solve the problem, obtained by viewing the events at slot boundaries in a different order. Suppose that the customer in service exits if its service is complete, and if so, the customer at the head of the queue (if any) is moved into the server before a possible arrival is added. In this view, a new customer still arrives to find the server busy with probability $\rho = \frac{7p}{2}$, but the residual service time is distributed over $\{1, 2, 3, 4, 5, 6\}$. The total service time of the customer in service has the same sampled lifetime distribution as before, with mean $E[L] = \frac{13}{3}$, but here, given L , γ is uniformly distributed over $\{1, \dots, L\}$, so $E[\gamma] = \frac{8}{3}$. Thus, by the PASTA property, $W = 3.5N + (\frac{7p}{2})(\frac{8}{3})$, where N is the mean number in the queue seen by an arrival. A customer waiting W slots for service can be seen in the queue for $W-1$ observation times, so by Little's law, $N = (W-1)p$. thus, $W = 3.5(W-1)p + (\frac{7p}{2})(\frac{8}{3})$, and solving for W yields $W = \frac{35p}{6-21p}$ for $0 \leq p < \frac{2}{7}$.

3.9. On stochastic ordering of sampled lifetimes

If $f_L \prec g_L$, then $\int_0^\infty \phi(x)f_L(x)dx > \int_0^\infty \phi(x)g_L(x)dx$ for any strictly decreasing positive function ϕ

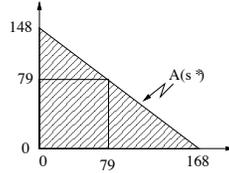
on \mathbb{R}_+ . Taking $\phi(x) = \frac{1}{x}$ yields that $1 = \int_0^\infty f(x)dx > \int_0^\infty g(x)dx = 1$, which is impossible.

3.10. Effective bandwidth, bufferless link

(a) $\alpha_1(s) = \frac{1}{s} \ln\left(\frac{e^{2s}-1}{2s}\right)$, $\alpha_2(s) = \frac{-\ln(1-s)}{s}$.

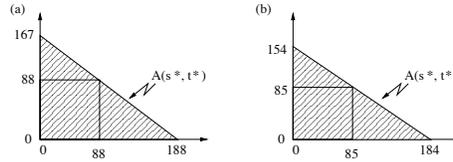
(b) $n=79.05$ (or round to $n = 79$)

(c) $\alpha_1(s^*) = 1.05$, $\alpha_2(s^*) = 1.19$, and $C_{eff}=177.0$. As one might expect, $\alpha_1(s^*)$ is somewhat smaller than $\alpha_2(s^*)$.



3.11. Effective bandwidth for a buffered link and long range dependent Gaussian traffic

(a) $n = 88.80$ (round down to $n=88$). $\alpha_1(s^*) = 1.0085$, $\alpha_2(s^*) = 1.1354$, $t^* = 31.6$, $C_{eff} = 190.38$.



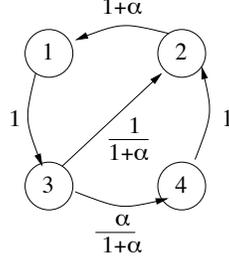
(b) $n = 85.33$ (round down to $n=85$). $\alpha_1(s^*) = 1.015$, $\alpha_2(s^*) = 1.1827$, $t^* = 22.7$, $C_{eff} = 187.53$. Due to the tighter overflow probability constraint, the effective bandwidths in part (b) are larger than those in part (a) and the effective bandwidth of the link is smaller. Interestingly, the critical time scale is smaller for part (b) than for part (a).

3.12. Extremal equivalent bandwidth for fixed mean and range

Intuitively, X should be as bursty as possible, suggesting the choice $P[X^* = 0] = P[X^* = 2] = 0.5$. Here's a proof. Fix $s > 0$. The function e^{sx} is convex in x over the interval $x \in [0, 2]$, with values 1 and e^{2s} at the endpoints. So $e^{sx} \leq 1 + \frac{x}{2}(e^{2s} - 1)$ for $x \in [0, 2]$. Plugging in X on each side, taking expectations, and using the constraint $E[X] = 1$ yields $E[e^{sX}] \leq \frac{1+e^{2s}}{2} = E[e^{sX^*}]$.

5.1. Time reversal of a simple continuous time Markov process

$$\begin{aligned}
 Q' &= \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3(1+\alpha)} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{\alpha}{3(1+\alpha)} \end{pmatrix}^{-1} Q^T \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3(1+\alpha)} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{\alpha}{3(1+\alpha)} \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1+\alpha & -(1+\alpha) & 0 & 0 \\ 0 & \frac{1}{1+\alpha} & -1 & \frac{\alpha}{1+\alpha} \\ 0 & 1 & 0 & -1 \end{pmatrix}
 \end{aligned}$$

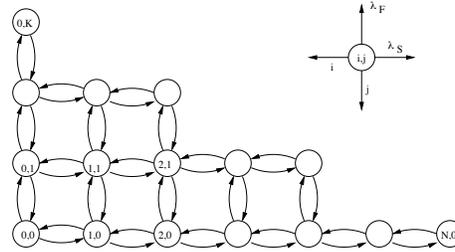


5.2. Reversibility of a three state Markov process

The process is reversible and $\pi = (\pi_1, \pi_2, \pi_3)$ is the equilibrium distribution, if and only if the detailed balance equations hold: $\pi_1 = a\pi_2$, $\pi_2 = b\pi_3$, $\pi_3 = c\pi_1$. The first two of the equations yield that $\pi_1 = ab\pi_3$, which can hold together with the third equation only if $abc = 1$. This condition is thus necessary. If $abc = 1$, we find that the detailed balance equations hold for the probability distribution proportional to $(ab, b, 1)$. Thus, the condition $abc = 1$ is also sufficient for reversibility.

5.3. A link with heterogeneous call requirements

(a)



(b) If there were infinitely many channels, the numbers of slow and fast connections would form independent $M/M/\infty$ systems, which is reversible, because each coordinate process is a one-dimensional, and hence reversible, Markov process, and two independent reversible processes are jointly reversible. The equilibrium distribution would be $\pi(i, j) = \frac{e^{-\lambda_S} \lambda_S^i}{i!} \frac{e^{-\lambda_F} \lambda_F^j}{j!}$. The process with the finite number of channels is obtained by truncation of the process with infinitely many channels, to the state space $\mathcal{S}^{K,L} = \{(i, j) \in \mathbb{Z}_+^2 : i + Lj \leq KL\}$, and is hence also reversible. The equilibrium distribution is $\pi^{K,L}(i, j) = \frac{\lambda_S^i \lambda_F^j}{i! j! Z(K,L)}$, where $Z(K, L)$ is selected to make $\pi^{K,L}$ sum to one.

(c) $B_S = \sum_{j=0}^L \pi^{K,L}((K-j)L, j)$, and $B_F = 1 - \frac{Z(K-1, L)}{Z(K, L)}$.

5.4. Time reversibility of an M/GI/1 processor sharing queue

(a) No, due to the memory in the service time variables. One Markov process is $\Gamma(t) = (N(t), S_1(t), \dots, S_{N(t)}(t))$, where $S_1(t) \geq \dots \geq S_{N(t)}(t) \geq 0$ and $S_i(t)$ denotes how much service the i th customer in the system received so far.

(b) Let $\mathbf{n} = (n_1, \dots, n_k)$ and $n = n_1 + \dots + n_k$. The nonzero off-diagonal rates are

$$q(\mathbf{n}, \mathbf{n}') = \begin{cases} \frac{n_i \mu_i}{n} & \text{if } \mathbf{n}' = (n_1, \dots, n_i - 1, n_{i+1} + 1, \dots, n_k) \\ \lambda & \text{if } \mathbf{n}' = (n_1 + 1, n_2, \dots, n_k) \\ \frac{n_k \mu_k}{n} & \text{if } \mathbf{n}' = (n_1, \dots, n_{k-1}, n_k - 1) \end{cases}$$

(c) The conjectured rate matrix for the time-reversed process is given by (only nonzero off-diagonal rates are indicated):

$$q'(\mathbf{n}', \mathbf{n}) = \begin{cases} \frac{(1+n_{i+1})\mu_{i+1}}{n} & \text{if } \mathbf{n}' = (n_1, \dots, n_i - 1, n_{i+1} + 1, \dots, n_k) \\ \frac{(1+n_1)\mu_1}{n+1} & \text{if } \mathbf{n}' = (n_1 + 1, n_2, \dots, n_k) \\ \lambda & \text{if } \mathbf{n}' = (n_1, \dots, n_{k-1}, n_k - 1) \end{cases}$$

To simultaneously prove the conjecture for π and q' , it suffices to check $\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n}') = \pi(\mathbf{n}')q'(\mathbf{n}', \mathbf{n})$ for $\mathbf{n} \neq \mathbf{n}'$. We find

$$\begin{aligned} \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} &= \begin{cases} \frac{a_{i+1}}{n_{i+1}+1} \frac{n_1}{a_i} = \frac{n_i \mu_i}{(n_{i+1}+1)\mu_{i+1}} & \text{if } \mathbf{n}' = (n_1, \dots, n_i - 1, n_{i+1} + 1, \dots, n_k) \\ \frac{(n+1)a_1 \rho}{n_1+1} = \frac{(n+1)\lambda}{\mu_1(n_1+1)} & \text{if } \mathbf{n}' = (n_1 + 1, n_2, \dots, n_k) \\ \frac{n_k}{\rho n a_k} = \frac{\mu_k n_k}{\lambda n} & \text{if } \mathbf{n}' = (n_1, \dots, n_{k-1}, n_k - 1) \end{cases} \\ &= \frac{q(\mathbf{n}, \mathbf{n}')}{q'(\mathbf{n}', \mathbf{n})} \end{aligned}$$

as required.

(d) The total service time for a customer has the same distribution in either system, namely the sum $Exp(\mu_1) + \dots + Exp(\mu_k)$. Hence, the random processes giving the number of customers in the system are statistically the same for the forward and reverse time systems.

5.5. A sequence of symmetric star shaped loss networks

(a) The set \mathcal{R} of routes is the set of subsets of $\{1, \dots, M\}$ of size two. Thus, $|\mathcal{R}| = \binom{M}{2}$. The process at time t has the form $X(t) = (X_r(t) : r \in \mathcal{R})$ with state space $\mathcal{S} = \{x \in \mathbb{Z}_+^{\mathcal{R}} : \sum_{r:i \in r} x_r \leq 5 \text{ for } 1 \leq i \leq M\}$ and transition rate matrix given by (only nonzero off diagonal rates are given):

$$\begin{aligned} q(x, x + e_r) &= \frac{4}{M-1} & \text{if } x, x + e_r \in \mathcal{S} \\ q(x, x - e_r) &= x_r & \text{if } x, x - e_r \in \mathcal{S} \end{aligned}$$

(b) The unthinned rate at a link ℓ is given by $\lambda_\ell = \sum_{r:\ell \in r} \nu_\ell = |\{r : r \in \ell\}| \times \frac{4}{M-1} = 4$, so for any M , $B_\ell \leq E[4, 5] = 0.19907$ and the call acceptance probability is greater than or equal to $(1 - 0.19907)^2 = 0.64149$.

If $M = 2$ then $B_\ell = E[4, 5]$ and the call acceptance probability is $1 - 0.19907 = 0.80093$.

(c) The reduced load approximation is given by the fixed point equations:

$$\widehat{\lambda}_\ell = \lambda(1 - \widehat{B}_\ell) \quad \text{and} \quad \widehat{B}_\ell = E[\widehat{\lambda}_\ell, C]$$

The equations do not depend on M , and the solution is $\widehat{B}_\ell = 0.1464$, leading to the estimate $(1 - \widehat{B}_\ell)^2 = 0.72857$ for the call acceptance probability. The fixed point approximation for this example, for both \widehat{B}_ℓ and call acceptance probability, can be shown to be exact in the limit $M \rightarrow \infty$. The numbers are summarized in the following table:

	B_ℓ	$P\{\text{route } r \text{ is free}\}$
bound	≤ 0.1999	≥ 0.6915
$M = 2$	0.199	0.811
$M = 3$	0.14438	0.745
$M = 4$	0.146	0.739
$M \rightarrow \infty$	0.146	0.728

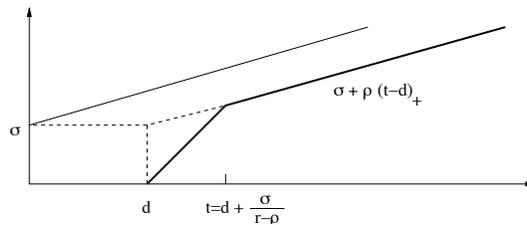
6.1. Deterministic delay constraints for two servers in series

- (a) The total arrival stream to the first queue is $(\sigma_1 + \sigma_2, \rho_1 + \rho_2)$ constrained, and the server is FIFO, so the delay is bounded above by $d_1 = \lceil \frac{\sigma_1 + \sigma_2}{C_1} \rceil$. ($d_1 = 2$ for the example.)
- (b) Stream 1 at the output of queue 1 is $(\sigma_1 + \rho_1 d_1, \rho_1)$ -upper constrained, because A_1 is (σ_1, ρ_1) -upper constrained and delay in the first queue is less than or equal to d_1 . ($\sigma = \sigma_1 + \rho_1 d_1 = 8$ for the example.)
- (c) By the example after Proposition 6.3.2, there is a bound on the delay of the lower priority stream for a FIFO server, constant service rate C , and (σ, ρ) constrained inputs. This yields:

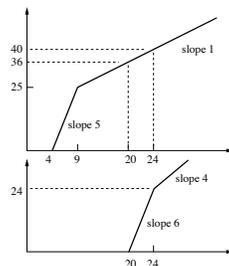
$$\text{delay for stream 1 in queue 2} \leq \frac{(\sigma_1 + \rho_1 d_1) + \sigma_3}{C_2 - \rho_3} = \frac{12}{3} = 4 \quad (\text{for the example})$$

6.2. Calculation of an acceptance region based on the SCED algorithm

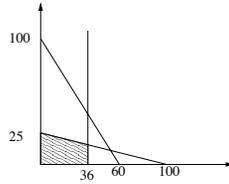
- (a) Let $g(t) = \sigma + \rho t$ for $\rho, \sigma > 0$. By the definition given in the notes, $g^*(0) = 0$ and for $t \geq 1$, $g^*(t) = \min\{g(s_1) + \dots + g(s_n) : n \geq 1, s_1, \dots, s_n \geq 1, s_1 + \dots + s_n = t\} = \min\{\sigma n + \rho t : n \geq 1\} = \sigma + \rho t$. Another approach to this problem is to use the definition of g^* in [5]. Yet another approach is to let $g_0(t) = I_{\{t \geq 1\}} g(t)$ and show that g_0 is the maximal subadditive function equal to 0 at 0, and less than or equal to g .
- (b) Before working out special cases, let's work out the general case. Let $g(t) = \sigma + \rho t$ and $f(t) = (t - d)_+$ where $\sigma > 0, D > 0, r > \rho > 0$. To find $g^* \star f$ by a graphical method, we use the formula $(g^* \star f)(t) = \min_{0 \leq s \leq t} \{g^*(s) + f(t - s)\}$. That is, $g^* \star f$ is the minimum of the functions obtained by considering $g^*(s) + f(t - s)$ as a function of t on the interval $t \geq s$ for each s fixed.



So, for the parameters at hand,



- (c) Since $n_1 g_1^* \star f_1 + n_2 g_2^* \star f_2$ is piecewise linear with initial value 0, this function is less than or equal to Ct for all $t \geq 0$ if inequality is true at breakpoints and at $+\infty$ (i.e. in the limit as $t \rightarrow \infty$). This requires $n_1(25, 36, 40, 1) + n_2(0, 0, 24, 4) \leq (900, 2000, 2400, 100)$, which simplifies to $n_1 \leq 36$ and $n_1 + 4n_2 \leq 100$.



6.3. Serve longer service priority with deterministically constrained arrival processes

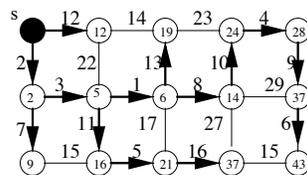
(a) The two queues combined is equivalent to a single queue system, with a work conserving server that serves up to C customers per slot. The total arrival stream is $(\sigma_1 + \sigma_2, \rho_1 + \rho_2)$ -upper constrained, so that the maximum combined carry over for a given slot t to the next satisfies $q(t) \leq \sigma_1 + \sigma_2$.

(b) By the same reason, the delay bound based on system busy periods can be applied to yield $d \leq \lfloor \frac{\sigma_1 + \sigma_2 - 1}{C - \rho_1 - \rho_2} \rfloor$.

(c) Under FIFO service within each stream and service to longer queue, each customer from stream 1 exits no later than it would exit from a system with the same arrival streams, FIFO service within each stream, and pure priority to type 2 customers, a case considered in the notes and class. The delay is thus bounded above by $\frac{\sigma_1 + \sigma_2}{C - \rho_2}$.

7.1. A shortest path problem

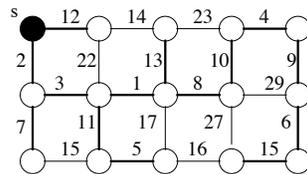
(a)



(b) Yes, because $D_j > D_i + d_{ij}$ for every edge ij not in the tree.

(c) Synchronous Bellman-Ford would take 8 iterations (on the 9th iteration no improvement would be found) because the most hops in a minimum cost path is 8.

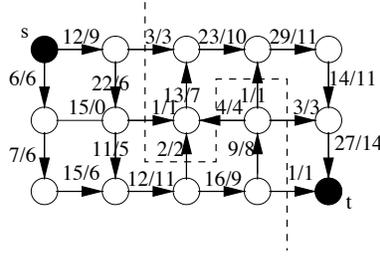
7.2. A minimum weight spanning tree problem



The MWST is unique because during execution of the Prim-Dijkstra algorithm there were no ties. (The sufficiency of this condition can be proved by a slight modification of the induction proof used to prove the correctness of the Prim-Dijkstra algorithm. Suppose the Prim-Dijkstra algorithm is executed. Prove by induction the following claim: The set of k edges found after k steps of the algorithm is a subset of *all* MWST's.)

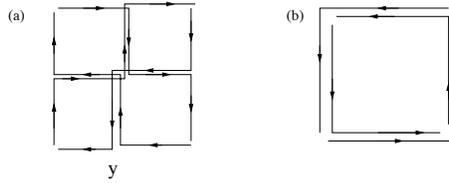
7.3. A maximum flow problem

The capacities, a maximum flow, and a minimum cut, are show. The maximum flow value is 15.



8.1. A flow deviation problem (Frank-Wolfe method)

- (a) Initially the 8 links directed counter clockwise along the perimeter of the network carry zero flow, and the other 16 links carry flow b . So $\text{cost}(f)=8b^2$.
- (b) Flow \bar{f} uses the 8 links not used initially, as shown.



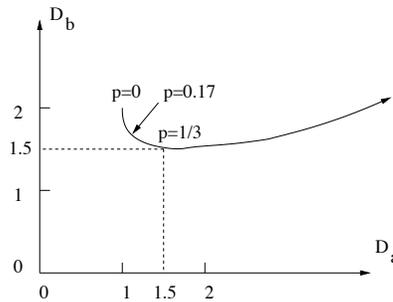
Thus, $\text{cost}(\alpha\bar{f} + (1 - \alpha)f) = 8(2\alpha b)^2 + 16((1 - \alpha)b)^2$. Minimizing over α , we find $\alpha^* = \frac{1}{3}$. Under the new flow, $f_1 = \alpha^*\bar{f} + (1 - \alpha^*)f$, all 24 links carry flow $\frac{2b}{3}$. The first derivative link length of all paths used is minimum. So f_1 is an optimal flow.

8.2. A simple routing problem in a queueing network

In equilibrium, the stations are independent and have the distribution of isolated M/M/1 queues, and the delay in an M/M/1 queue with arrival rate λ and departure rate μ is $\frac{1}{\mu-\lambda}$. Therefore,

$$D_a = \frac{1}{2 - (1 + p)} \quad \text{and} \quad D_b(p) = p \left(\frac{1}{2 - (1 + p)} \right) + (1 - p) \left(\frac{1}{2 - (1 - p)} + \frac{1}{2 - (1 - p)} \right)$$

(b)



- (c) $p = 0$ minimizes D_a , because D_a is increasing in p .
- (d) Setting $D'_b(p) = 0$ yields $p = \frac{1}{3}$.
- (e) Setting $D'_a(p) + D'_b(p) = 0$ yields $p \approx 0.17$. Only one iteration of the flow deviation algorithm is necessary because the only degree of freedom is the split for flow b between two paths. The line

search for flow deviation solves the problem in one iteration.

(f) The solution is $p = \frac{1}{3}$, because D_a is increasing in p , D_b is decreasing in p , and $D_a(\frac{1}{3}) = D_b(\frac{1}{3})$. (It is a coincidence that the answers to (d) and (f) are the same.)

8.3. A joint routing and congestion control problem

(a) Since $D'_i(F) = F$, the price for a link is equal to the flow on the link. For example, path 124 has price $2a + d$. The optimality conditions are:

$$\begin{aligned} 2a + d &\geq \frac{\beta_{14}}{a + b} && \text{with equality if } a > 0 \\ 2b + c &\geq \frac{\beta_{14}}{a + b} && \text{with equality if } b > 0 \\ 2c + b &\geq \frac{\beta_{23}}{c + d} && \text{with equality if } c > 0 \\ a + 2d &\geq \frac{\beta_{23}}{c + d} && \text{with equality if } d > 0 \end{aligned}$$

The first two conditions are for o-d pair $w = (1, 2)$ and the last two are for o-d pair $w = (2, 3)$.

(b) If $a, b, c, d > 0$ the equality constraints hold in all four conditions. The first two conditions yield $2a + d = 2b + c$ or, equivalently, $2a - 2b = c - d$. The last two yield $2c + b = a + 2d$ or, equivalently, $2a - 2b = 4c - 4d$. Thus, $c - d = 4c - 4d$ or, equivalently, $c = d$, and hence also $a = b$. (c) We seek a solution such that all four flows are nonzero, so that part (b) applies. Replacing b by a and d by c , and using $\beta_{14} = 66$ and $\beta_{23} = 130$, the optimality conditions become

$$\begin{aligned} 2a + c &= \frac{66}{2a} \\ 2c + a &= \frac{130}{2c} \end{aligned}$$

or $a = b = 3$ and $c = d = 5$. To double check, note that the two routes for $w = (1, 4)$ have price 11 each, which is also the marginal value for route $(1, 4)$, and the two routes for $w = (2, 3)$ have price 13 each, which is also the marginal value for route $(2, 3)$.

8.4. Joint routing and congestion control with hard link constraints

(a) The primal variables should be feasible: $a, b, c, d \geq 0$ and $a \leq C_1, b + c \leq C_2, d \leq C_3$. The dual variables should satisfy the positivity and complementary slackness condition: $p_i \geq 0$ with equality if link i is not saturated. It is easy to see that the optimal flow will saturate all three links, so that nonzero prices are permitted. Finally, since $U'_1(x_1) = x_1^{-1/2}$ and $U'_2(x_2) = \ln(x_2)$, the remaining conditions are:

$$\begin{aligned} (a + b)^{-1/2} &\leq p_1 && \text{with equality if } a > 0 \\ (a + b)^{-1/2} &\leq p_2 && \text{with equality if } b > 0 \\ \frac{1}{c + d} &\leq p_2 && \text{with equality if } c > 0 \\ \frac{1}{c + d} &\leq p_3 && \text{with equality if } d > 0 \end{aligned}$$

(b) Clearly we can set $a = d = 8$ and $c = 8 - b$. It remains to find b . The optimality conditions become

$$\begin{aligned} (8 + b)^{-1/2} &\leq p_2 && \text{with equality if } b > 0 \\ \frac{1}{16 - b} &\leq p_2 && \text{with equality if } b < 8 \end{aligned}$$

Since $(8+b)^{-1/2} > \frac{1}{16-b}$ over the entire range $0 \leq b \leq 8$, the optimal choice of b is $b = 8$. This yields the assignment $(a, b, c, d) = (8, 8, 0, 8)$, price vector $(p_1, p_2, p_3) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{8})$, and maximum value $2 + \ln(8)$.

8.5. Sufficiency of the optimality condition for hard link constraints

Since $L(x, p)$ is concave in x , for any ϵ with $0 < \epsilon < 1$,

$$L(x, p) - L(x^*, p) \leq \frac{1}{\epsilon} \{L(x^* + \epsilon(x - x^*), p) - L(x^*, p)\}$$

Taking the limit as $\epsilon \rightarrow 0$ yields

$$\begin{aligned} L(x, p) - L(x^*, p) &\leq \sum_r U'_r(x_r^*)(x_r - x_r^*) - \sum_l p_l \left(\sum_{r:l \in r} x_r - x_r^* \right) \\ &= \sum_r (x_r - x_r^*) \left\{ U'_r(x_r^*) - \sum_{l \in r} p_l \right\} \leq 0 \end{aligned} \quad (10.19)$$

where the final inequality follows from the fact that for each r , the quantity in braces in (10.19) is less than or equal to zero, with equality if $x_r^* > 0$.

(b) Note that $\sum_l p_l (C_l - \sum_{r:l \in r} x_r^*) = 0$ and if $Ax \leq C$, then $\sum_l p_l (C_l - \sum_{r:l \in r} x_r) \geq 0$. So if $x \geq 0$ and $Ax \leq C$,

$$\sum_r U(x_r^*) = L(x^*, p) \geq L(x, p) \geq \sum_r U(x_r).$$

8.6. Fair flow allocation with hard constrained links

(a) By inspection, $x_{max-min} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

(b) (proportional fairness) Let p_l denote the price for link l . Seek a solution to the equations

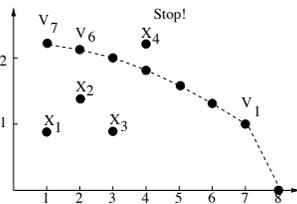
$$\begin{aligned} x_1 &= \frac{1}{p_1 + p_2 + p_3} & x_2 &= \frac{1}{p_1 + p_2} & x_3 &= \frac{1}{p_1} & x_4 &= \frac{1}{p_2 + p_3} \\ x_1 + x_2 + x_3 &\leq 1, \text{ with equality if } p_1 > 0 \\ x_1 + x_2 + x_4 &\leq 1, \text{ with equality if } p_2 > 0 \\ x_1 + x_4 &\leq 1, \text{ with equality if } p_3 > 0 \end{aligned}$$

Clearly $x_1 + x_4 < 1$, so that $p_3 = 0$. Also, links 1 and 2 will be full, so that $x_3 = x_4$. But $x_3 = \frac{1}{p_1}$ and $x_4 = \frac{1}{p_2 + p_3}$, so that $p_1 = p_2$. Finally, use $\frac{1}{2p_1} + \frac{1}{2p_1} + \frac{1}{p_1}$ to get $p_1 = p_2 = 2$, yielding $x_{pf} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$. Flows 1 and 2 use paths with price $p_1 + p_2 = 4$ and each have rate $\frac{1}{4}$. Flows 3 and 4 use paths with price $p_1 = p_2 = 2$ and each have rate $\frac{1}{2}$.

9.1. Illustration of dynamic programming – a stopping time problem

(a) Consider the game for a possible $n + 1$ observations. After seeing X_1 , the player can either stop and receive reward X_1 , or continue and receive expected reward V_n . Thus, $V_{n+1} = E[X_1 \vee V_n] = \int_0^\infty (x \vee V_n) e^{-x} dx = \int_0^{V_n} V_n e^{-x} dx + \int_{V_n}^\infty x e^{-x} dx = V_n + e^{-V_n}$.

(b) The optimal policy is threshold type. For $1 \leq k \leq n - 1$, the player should stop after observing X_k if $X_k \geq V_{n-k}$. The rule for $n = 8$ is pictured.



n	V_n	$1 + \frac{1}{2} + \dots + \frac{1}{n}$
1	1.00000	1.00000
2	1.36788	1.50000
3	1.62253	1.83333
4	1.81993	2.08333
5	1.98196	2.28333
(c) 6	2.11976	2.45000
7	2.23982	2.59286
8	2.34630	2.71786
9	2.44202	2.82897
10	2.52901	2.92897
20	3.12883	3.59774
30	3.49753	3.99499

9.2. Comparison of four ways to share two servers

(a) *System 1:* Each of the two subsystems is an M/M/1 queue, so that N_1 is twice the mean number in an M/M/1 queue with arrival rate λ and departure rate μ : $N_1 = \frac{2\rho}{1-\rho}$.

System 2: Each of the two subsystems is G/M/1 where the interarrival distribution is the same as the sum of two exponential random variables with parameter 2λ . We will analyze such a G/M/1 queue. The Laplace transform of the interarrival distribution is $A^*(s) = (\int_0^\infty e^{-st} 2\lambda e^{-2\lambda t} dt)^2 = (\frac{2\lambda}{s+2\lambda})^2$. We seek the solution σ in the range $0 \leq \sigma < 1$ of the equation $\sigma = A^*(\mu - \mu\sigma)$, or $\sigma^3 - 2(1+2\rho)\sigma^2 + (1+4\rho(1+\rho))\sigma - 4\rho^2 = 0$. Since $\sigma = 1$ is a solution, this equation can be written as $(\sigma - 1)(\sigma^2 - (1+4\rho)\sigma + 4\rho^2) = 0$ which has solutions $\sigma = 1$ and $\sigma = \frac{(1+4\rho) \pm \sqrt{1+8\rho}}{2}$. Thus, there is a solution $\sigma \in [0, 1)$ if and only if $0 \leq \rho < 1$, and it is given by $\sigma = \frac{1+4\rho - \sqrt{1+8\rho}}{2}$. Therefore, $N_2 = \frac{2\rho}{1-\sigma} = \frac{4\rho}{1-4\rho + \sqrt{1+8\rho}}$.

System 3 The third system is an M/M/2 queueing system. The number in the system is a birth-death Markov process with arrival rates $\lambda_k = 2\lambda$ for all $k \geq 0$ and death rates $\mu_k = (k \wedge 2)\mu$ for $k \geq 1$. Let $\rho = \frac{\lambda}{\mu}$. The usual solution method for birth-death processes yields that $p_1 = \frac{2\lambda}{\mu} p_0$ and $p_k = 2\rho^k p_0$ for $k \geq 1$. The process is positive recurrent if and only if $\rho < 1$, and for such ρ we find $p_0 = (1 + 2(\rho + \rho^2 + \dots)) = \frac{1-\rho}{1-\rho^2}$ and $N_3 = \frac{2\rho}{1-\rho^2}$, which is smaller than N_1 by a factor $1 + \rho$.

(b) For $0 < \rho < 1$, $N_3 < N_4 < N_2 < N_1$. Here is the justification. First, we argue that $N_3 < N_4$, using the idea of stochastic domination. Indeed, let there be three independent Poisson streams: an arrival stream A with rate 2λ , and for $i = 1, 2$ a potential departure stream D_i with departure rate μ_i . Consider systems 3 and 4 running, using these same three processes. If stream D_i has a jump at time t , then there is a departure from system 3 if queue i is not empty, and there is a departure from system 4 if the entire system is not empty. By induction on the number of events (arrival or potential departure events) we see that the number of customers in system 3 is always less than or equal to the number of customers in system 4, and is occasionally strictly less. Thus, $N_3 < N_4$. We argue using dynamic programming that system 4, using send to the shorter queue, is using the

so we see that indeed $u = 0$ is optimal when $x = K$. Thus, using $g(x, u) = c\lambda_o I_{\{x=K\}} + r\mu I_{\{x=0\}}$, we have that

$$V_{n+1}(x) = g(x) + \frac{\beta}{\gamma} [\mu V_n((x-1)_+) + \lambda_o V_n((x+1) \wedge K) + \lambda \min \{V_n(x), V_n((x+1) \wedge K)\}]$$

(c) When there are n steps-to-go and the current state is x , an optimal control is given by

$$u_n(x) = \begin{cases} 0 & \text{if } V_n(x) \leq V_n((x+1) \wedge K) \\ 1 & \text{if } V_n(x) > V_n((x+1) \wedge K) \end{cases}$$

(d) Let us prove that for each $n \geq 1$ there exists a threshold τ_n such that $u_n(x) = I_{\{x \leq \tau_n\}}$. It is enough to show that the following is true for each $n \geq 1$:

(a) V_n is convex, i.e. $V_n(1) - V_n(0) \leq V_n(2) - V_n(1) \leq \dots \leq V_n(K) - V_n(K-1)$

(b) $-\frac{r\gamma}{\beta} \leq V_n(1) - V_n(0)$

(c) $V_n(K) - V_n(K-1) \leq \frac{c\gamma}{\beta}$

To complete the proof, we prove (a)-(c) by induction on n . The details are very similar to those for the $M_{\text{controlled}}/M/1$ example in the notes, and are omitted.

(e) For an exact formulation, we need to describe a state space. If FCFS service order is assumed, it is not enough to know the number of customers of each type in the system, so that the state space would be the set of sequences from the alphabet $\{1, 2\}$ with length between 0 and K . We expect the optimal control to have some monotonicity properties, but it would be hard to describe the control because the state space is so complex. If instead, the service order were pure preemptive resume priority to customers of one type, or processor sharing, then the state space could be taken to be $\{(n_1, n_2) : n_1 \geq 0, n_2 \geq 0, \text{ and } n_1 + n_2 \leq K\}$. We expect the optimal control to have a switching curve structure.

9.4. Control dependent cost

(a)

$$\begin{aligned} E_x \int_0^{t_n} g(X(t), u(t)) e^{-\alpha t} dt &= \sum_{k=0}^{n-1} E_x \int_{t_k}^{t_{k+1}} g(X(t), u(t)) e^{-\alpha t} dt \\ &= \sum_{k=0}^{n-1} E_x \int_{t_k}^{t_{k+1}} g(X(t_k), w_k) e^{-\alpha t} dt \\ &= \sum_{k=0}^{n-1} E_x [g(X(t_k), w_k) \int_{t_k}^{t_{k+1}} e^{-\alpha t} dt] \\ &= \sum_{k=0}^{n-1} E_x [g(X(t_k), w_k)] E_x [\int_{t_k}^{t_{k+1}} e^{-\alpha t} dt] \\ &= \frac{1-\beta}{\alpha} E_x \sum_{k=0}^{n-1} \beta^k g(X(t_k), w_k) \end{aligned}$$

where we used the fact that

$$E_x \left[\int_{t_k}^{t_{k+1}} e^{-\alpha t} dt \right] = E_x \left[\frac{1}{\alpha} (e^{-\alpha t_k} - e^{-\alpha t_{k+1}}) \right] = \frac{1}{\alpha} [\beta^k - \beta^{k+1}] = \frac{(1-\beta)\beta^k}{\alpha}.$$

(b) The only change is that g should be added to the cost before the minimization over u , yielding:

$$V_{n+1}(x) = \inf_{u \in \mathcal{U}} \left[g(x, u) + \beta \sum_y p_{xy}(u) V_n(y) \right].$$

9.5. Optimal control of a server

(a) We use $\mathcal{S} = \mathbb{Z}_+^2$, where state $x = (x_1, x_2)$ denotes x_i customers in station i , for $i = 1, 2$. Let $A_1(x_1, x_2) = (x_1 + 1, x_2)$ and $D_1(x_1, x_2) = ((x_1 - 1)_+, x_2)$, and define A_2 and D_2 similarly. Then

$$p_{x,y}(u) = \frac{1}{\gamma} \sum_{i=1}^2 (\lambda_i I_{\{y=A_i(x)\}} + m_i(u) I_{\{y=D_i(x)\}})$$

The CDF F corresponds to the exponential distribution with parameter $\gamma = \lambda_1 + \lambda_2 + \mu_0 + \mu_1 + \mu_2$, and the instantaneous cost function is $g(x) = x_1 + x_2$.

(b) The backwards equation of dynamic programming becomes

$$V_{n+1}(x) = g(x) + \min_{0 \leq u \leq 1} \frac{\beta}{\gamma} \sum_{i=1}^2 (\lambda_i V_n(A_i(x)) + m_i(u) V_n(D_i(x)))$$

or, after plugging in the optimal value for u ,

$$V_{n+1}(x) = g(x) + \frac{\beta}{\gamma} \left[\sum_{i=1}^2 (\lambda_i V_n(A_i(x)) + \mu_i(u) V_n(D_i(x))) + \mu_0 \min\{V_n(D_1(x)), V_n(D_2(x))\} \right]$$

with the initial condition $V_0 \equiv 0$.

(c)

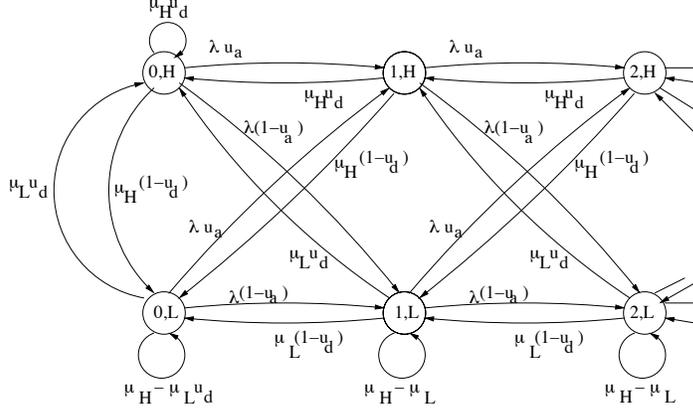
$$u_n^*(x) = \begin{cases} 1 & \text{if } V_n(D_1x) \leq V_n(D_2x) \\ 0 & \text{else.} \end{cases} \quad (10.20)$$

(d) We conjecture that the service rate μ_0 should be allocated to the station with the longer queue. That is, $u_n^*(x) = I_{\{x_1 \geq x_2\}}$. Equivalently, $V(x_1, x_2) \leq V(x_1 - 1, x_2 + 1)$ whenever $0 \leq x_1 \leq x_2$ and $V(x_1, x_2) \leq V(x_1 + 1, x_2 - 1)$ whenever $0 \leq x_2 \leq x_1$.

9.6. A dynamic server rate control problem with switching costs

(a) We use the state space $\mathcal{S} = \{(l, \theta) : l \in \mathbb{Z}_+, \theta \in \{H, L\}\}$, where for a given state $x = (l, \theta)$, l denotes the number of customers in the system and θ denotes the state of the server. Let $\gamma = \lambda + \mu_H$, which is the maximum event rate that we shall use. (Another natural choice would be $\gamma = \lambda + \mu_L + \mu_H$, and then the self loop transition probabilities would be increased in the one step transition probabilities.) We take the control values $u = (u_a, u_d)$ to be in the set $\mathcal{U} = [0, 1]^2$, where u_a is the probability the server is in the high state after the next event, given the next event is an arrival, and u_d is the probability the server is in the high state after the next event, given the next event is a departure. The following diagram gives the transition probabilities (the quantities shown should be divided by γ).²

²Variations of these equations are possible, depending on the exact assumptions. We chose to allow a server state change at potential departure times, even if the potential departures are not used.



The total costs can be described as follows.

$$\text{cost due to use of high service rate and queueing} = E_x \left[\int_0^T \{c_H I_{\theta_t=H}\} + c_W l_t \} e^{-\alpha t} dt \right]$$

$$\begin{aligned} \text{cost due to switching} &= E_x \left[\sum_{t \leq T} c_S I_{\{\theta_t=L, \theta_t=H\}} e^{-\alpha t} \right] \\ &= E_x \left[\int_0^T c_S I_{\{\theta_t=L\}} [\lambda u_a(t) + \mu_L u_d(t)] e^{-\alpha t} dt \right] \end{aligned}$$

Thus, the cost is captured by taking

$$g(x, u) = g((l, \theta), (u_a, u_d)) = c_H I_{\{\theta=H\}} + c_W l + c_S I_{\{\theta=L\}} [\lambda u_a + \mu_L u_d]$$

(b) We first consider $V_{n+1}(x)$ for states $x = (l, \theta)$ with $\theta = H$. Since $g((l, H), u) = c_H + c_W l$, which doesn't depend on u , the backwards recursion is:

$$\begin{aligned} V_{n+1}(l, H) = c_H + c_W l + \beta \min_{u \in [0,1]^2} & \left\{ \frac{\lambda u_a}{\gamma} V_n(l+1, H) + \frac{\lambda(1-u_a)}{\gamma} V_n(l+1, L) \right. \\ & \left. + \frac{\mu_H u_d}{\gamma} V_n((l-1)_+, H) + \frac{\mu_H(1-u_d)}{\gamma} V_n((l-1)_+, L) \right\} \end{aligned}$$

The analogous equation for $\theta = L$ is

$$\begin{aligned} V_{n+1}(l, L) = \min_{u \in [0,1]^2} & c_W l + c_S \{\lambda u_a + \mu_L u_d\} + \beta \left\{ \frac{\lambda u_a}{\gamma} V_n(l+1, H) + \frac{\lambda(1-u_a)}{\gamma} V_n(l+1, L) \right. \\ & \left. + \frac{\mu_L u_d}{\gamma} V_n((l-1)_+, H) + \frac{\mu_L(1-u_d)}{\gamma} V_n((l-1)_+, L) + \frac{\mu_H - \mu_L}{\gamma} V_n(l, L) \right\} \end{aligned}$$

Solving for u in the above equations and setting $\delta = \frac{c_S \gamma}{\beta}$ yields the following simpler version of the

dynamic programming backwards recursion:

$$V_{n+1}(l, H) = c_H + c_W l + \frac{\beta\lambda}{\gamma} \min\{V_n(l+1, H), V_n(l+1, L)\} \\ + \frac{\beta\mu_H}{\gamma} \min\{V_n((l-1)_+, H), V_n((l-1)_+, L)\} \quad (10.21)$$

$$V_{n+1}(l, L) = c_W l + \frac{\beta\lambda}{\gamma} \min\{\delta + V_n(l+1, H), V_n(l+1, L)\} \\ + \frac{\beta\mu_L}{\gamma} \min\{\delta + V_n((l-1)_+, H), V_n((l-1)_+, L)\} + \frac{\beta(\mu_H - \mu_L)}{\gamma} V_n(l, L) \quad (10.22)$$

The optimal controls can be succinctly described as follows. If the current switch state is θ , and if there will be l' customers in the system after the next arrival or potential departure, then the optimal server state after such arrival or potential departure is given by:

$$\begin{aligned} \text{If } \theta = H \text{ then: } & V_n(l', L) > V_n(l', H) \quad \Leftrightarrow \text{ new server state H} \\ \text{If } \theta = L \text{ then: } & V_n(l', L) > V_n(l', H) + \delta \quad \Leftrightarrow \text{ new server state H} \end{aligned} \quad (10.23)$$

The function $V_n(l, \theta)$ represents the cost-to-go for initial state (l, θ) , with the understanding that it is not possible to switch the server state until the first event time. Let $W_n(l, \theta)$ denote the cost-to-go, given the initial state is (l, θ) , but assuming that the server state can be changed at time zero. Then the dynamic programming equations become:³

$$W_n(l, \theta) = \min\{V_n(l, L), V_n(l, H) + \delta I_{\{\theta=L\}}\} \quad (10.24)$$

$$V_{n+1}(l, \theta) = c_H I_{\{\theta=H\}} + c_W l + \frac{\beta\lambda}{\gamma} W_n(l+1, \theta) + \frac{\beta\mu_L}{\gamma} W_n((l-1)_+, \theta) \\ + \frac{\beta(\mu_H - \mu_L)}{\gamma} \{I_{\{\theta=H\}} W_n(l-1, H) + I_{\{\theta=L\}} W_n(l, L)\} \quad (10.25)$$

with the initial conditions $V_0 \equiv 0$ and $W_n \equiv 0$.

In the special case that there is no switching cost, $\delta = 0$. In that case, $W_n(l, \theta)$ does not depend on θ , so we write it as $W_n(l)$. Then the dynamic programming equations become:

$$W_n(l) = \min\{V_n(l, L), V_n(l, H)\} \quad (10.26)$$

$$V_{n+1}(l, \theta) = c_H I_{\{\theta=H\}} + c_W l + \frac{\beta\lambda}{\gamma} W_n(l+1) + \frac{\beta\mu_L}{\gamma} W_n((l-1)_+) \\ + \frac{\beta(\mu_H - \mu_L)}{\gamma} \{I_{\{\theta=H\}} W_n(l-1) + I_{\{\theta=L\}} W_n(l)\} \quad (10.27)$$

In the case of no switching costs, it is rather simple to show by induction on n that W_n is a convex, nondecreasing function on \mathbb{Z}_+ and V_n satisfies properties 1-5 in the original solutions. As noted in the original solutions, property 3 for V_n implies the conjectured threshold structure.

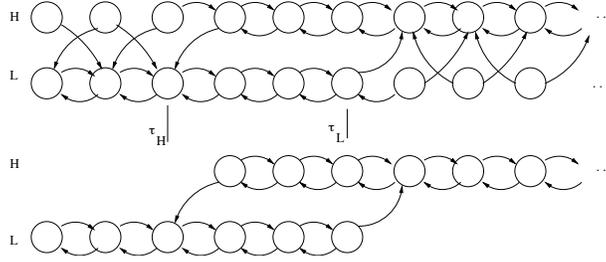
(c) We conjecture that the optimal control has a threshold behavior, because when there are more customers in the system, it is more valuable to quickly serve the customer in the server, thereby reducing the waiting times of all remaining customers. Since there is a switching cost,

³In this version, we allow a change of server state after any event. Thus, the functions V_n are slightly different than the ones above.

however, it is better to be a bit reluctant to switch, because excessive switching is costly. This suggests that the threshold for switching from H to L should be smaller than the threshold for switching from L to H . Specifically, we expect there are two thresholds, τ_H and τ_L , with $\tau_H \leq \tau_L$, so that the optimal control law (10.23) becomes

$$l' > \tau_\theta \Leftrightarrow \text{new server state } H \tag{10.28}$$

This leads to the nonzero transition probabilities shown. The transient states are included in the first diagram and are omitted in the second.



We haven't found a proof of the conjectured threshold structure in general, but we here sketch a proof in the case of zero switching cost ($\delta = 0$). Consider the following properties of a function V on $\mathbb{Z}_+ \times \{L, H\}$:

1. $V(l, \theta)$ is nondecreasing in l for $\theta = L$ and for $\theta = H$.
2. $V(l, \theta)$ is convex in l for $\theta = L$ and for $\theta = H$.
3. $V(l + 1, L) - V(l + 1, H) - V(l, L) + V(l, H) \geq 0$ for $l \geq 0$.
4. $V(l + 2, H) - V(l + 1, H) - V(l + 1, L) + V(l, L) \geq 0$ for $l \geq 0$.
5. $V(l + 2, L) - V(l + 1, L) - V(l + 1, H) + V(l, H) \geq 0$ for $l \geq 0$.

We write property 1.H to denote property 1 for $\theta = H$. Properties 1.L, 2.H, and 2.L are defined analogously. The following figure gives a graphical representation of the five properties.

Property:	1.L	1.H	2.L	2.H	3	4	5
H		+		+	+	-	+
L	-	+	+	-	-	+	-

For each property, the figure indicates the type of linear combinations of values of V which should be nonnegative. The collection of properties 1-5 are nearly symmetric in H and L . Property 5 is obtained by swapping L and H in property 4. Thus, property 3 is the only part of properties 1-5 that is not symmetric in L and H .

Properties 3 and 4 imply property 2 (both 2.L and 2.H). This can be seen graphically by sliding the diagram for property 4 to the left to overlap the diagram for property 3, and adding. Similarly, properties 2.L and 3 imply property 5. Thus, properties 3 and 4 together imply properties 2 and 5. So to prove a function has properties 1-5, it suffices to prove it has properties 1, 3, and 4.

Property 3 is the one connected to the threshold structure. Another way to state property 3 is that $V(l, H) - V(l, L)$ is nonincreasing in l . That is, as l increases, given there are l customers in the system, it becomes increasingly preferable to be in state H . More specifically, if V_n satisfies property 3, and if $\tau_L = \max\{l : V(l, H) - V(l, L) \geq 0\}$ and $\tau_H = \max\{l : V(l, H) + \delta - V(l, L) \geq 0\}$, then (10.23) is equivalent to (10.28).

It remains to show that V_n has properties 1-5 for all $n \geq 0$. To do so, it can be proved by induction on n that V_n has properties 1-5 for all $n \geq 0$, and W_n is convex and nondecreasing on \mathbb{Z}_+ . For the base case, observe that the function V_0 given by $V_0 \equiv 0$ has properties 1-5.

For the general induction step, it can be easily shown that if V_n has properties 1-5, then W_n is convex, nondecreasing. And given that W_n is convex, nondecreasing, it can be shown that V_{n+1} has properties 1-5. As mentioned above, it suffices to establish that V_{n+1} has properties 1,3, and 4. Unfortunately, this approach doesn't seem to work in case $\delta > 0$.

(d) First, if $c_S = 0$ then the cost-to-go does not depend on the server state, and in particular the thresholds should be equal: $\tau_L = \tau_H$. Suppose a single customer is served, with no other customers waiting or arriving. If the server is put into the high state, and if t_H denotes the time the customer departs, then the average cost is $E[\int_0^\infty e^{-\alpha t} dt](c_H + c_W) = E[\frac{1 - e^{-\alpha t_H}}{\alpha}](c_H + c_W) = \frac{c_H + c_W}{\alpha + \mu_H}$. Similarly, if the server is put into the low state during the service interval, then the cost is $\frac{c_W}{\alpha + \mu_L}$. The given inequality implies that it costs less to use the high rate, even if there is only one customer in the system over all time. But using the high rate for a given server helps decrease the costs for any other customers that might be in the system, so that it is optimal to always use the high rate (when the system is not empty), under the conditions given in the problem.

Bibliography

- [1] S. Asmussen, *Applied Probability and Queues*, Wiley, New York, 1987.
- [2] D. Bertsekas and R.G. Gallager, *Data Networks, 2nd ed.*, Prentice-Hall, Englewood Cliffs, New Jersey, 1992.
- [3] D.D. Botvich and N. Duffield, “Large deviations, the shape of the loss curve, and economies of scale in large multiplexers,” *Queueing Systems*, vol. 20, 293-320, 1995.
- [4] C.-S. Chang, “Stability, queue length, and delay of deterministic and stochastic queueing networks,” *IEEE Trans. on Automatic Control*, vol. 39, no. 5, pp. 913-931, May 1994.
- [5] C.-S. Chang, *Performance Guarantees in Communication Networks*, Springer, New York, 2001.
- [6] J. Cohen and F. Kelly, “A paradox of congestion control in queueing networks,” *Journal of Applied Probability*, vol. 27, 1990.
- [7] C. Courcoubetis and R. Weber, “Buffer overflow asymptotics for a switch handling many traffic sources,” *J. Applied Probability*, vol. 33, no. 3, 886-903, 1996.
- [8] R.L. Cruz, “A calculus for network delay, part I: network elements in isolation,” *IEEE Transactions on Information Theory*, vol. 37, pp. 114–131, Jan. 1991.
- [9] R.L. Cruz, “A calculus for network delay, part II: network analysis,” *IEEE Transactions on Information Theory*, vol. 37, pp. 132–141, Jan. 1991.
- [10] R.L. Cruz, “Service burstiness and dynamic burstiness measures: a framework,” *Journal of High Speed Networks*, vol. 1, no. 2, pp. 105-127, 1992.
- [11] G. de Veciana, G. Kesidis, and J. Walrand, “Resource management in wide-area ATM networks using effective bandwidths”, *IEEE J. Selected Areas of Communications*, vol. 13, no. 6, 1081-1090, 1995.
- [12] J. Edmonds, “Submodular functions, matroids and certain polyhedra,” *Proc. Calgary Int. Conf. Combinatorial Structures and Applications*, Calgary, Alta, June 1969, pp. 6987.
- [13] A. Elwalid, D. Mitra, and R.H. Wentworth, “A new approach for allocating buffers and bandwidth to heterogeneous regulated traffic in an ATM node,” *IEEE J. Selected Areas of Communications*, vol. 13, no. 6, pp. 991-1003, 1995.
- [14] F. G. Foster, “On the stochastic matrices associated with certain queueing processes,” *Ann. Math Statist.*, Vol. 24, pp. 355-360, 1953.

- [15] L. Fratta, M. Gerla, and L. Kleinrock, "The flow deviation method: an approach to store and forward communication network design." *Networks*, vol. 3, pp. 97-133, 1973.
- [16] R.G. Gallager, P.A. Humblet, and P.M. Spira, "A distributed algorithm for minimum-weight spanning trees," *ACM Trans. Programming Languages and Systems*, 1983.
- [17] G.L. Georgiadis, R. Guérin, V. Peris, and K.N. Sivrajan, "Efficient network QOS provisioning based on per node traffic shaping," *IEEE/ACM Transactions on Networking*, vol. 4, no. 4, pp. 482-501, August 1996.
- [18] G.R. Grimmett and D.R. Stirzaker, *Probability and Random Processes*, 3rd Ed., Oxford University Press, Oxford and New York, 2001.
- [19] B. Hajek, "Optimal control of two interacting service stations," *IEEE Trans. Automatic Control*, Vol. 29, June 1984, pp. 491-499.
- [20] J.Y. Hui, "Resource allocation for broadband networks," *IEEE J. Selected Areas of Communications*, vol. 13, no. 6, 1598-1608.
- [21] J.Y. Hui *Switching an traffic theory for integrated broadband networks*, Kluwer, Boston, 1990.
- [22] J.Y. Hui and E. Karasan, "A thermodynamic theory of broadband networks with application to dynamic routing," *IEEE J. Selected Areas of Communications*, vol. 13, no. 6, 991-1003, 1995.
- [23] F.P. Kelly, "Notes on effective bandwidth" In *Stochastic Networks Theory and Applications*, F.P. Kelly, S. Zachary and I. Ziedins, Eds., Oxford Science, pp. 141-168, 1996.
- [24] F. P. Kelly, "Charging and rate control for elastic traffic," *European Transactions on Telecommunications*, Vol. 8, (1997), pp. 33-37.
- [25] F. P. Kelly, A.K. Maulloo, and D.K.H. Tan, "Rate control in communication networks: shadow prices, proportional fairness and stability," *Journal of the Operational Research Society* 49 (1998), 237-252.
- [26] J.F.C. Kingman, "Some inequalities for the queue GI/G/1," *Biometrika*, vol. 49, no. 3 and 4, p. 315-324, 1962.
- [27] L. Kleinrock, *Queueing Systems, Vol. 2: Computer Applications*, Wiley, 1976.
- [28] P.R. Kumar and S.P. Meyn, "Stability of queueing networks and scheduling policies," *IEEE Trans. Automatic Control*, vol. 40, 1995, pp. 251-260.
- [29] N. McKeown, A. Mekkittikul, V. Anantharam, and J. Walrand, "Achieving 100% throughput in an input-queued switch," *IEEE Transactions on Communications*, vol. 47, no. 8, 1999.
- [30] S.P. Meyn and R.L. Tweedie, *Markov chains and stochastic stability*, Springer-Verlag, London, 1993.
- [31] J. Mo and J. Walrand, "Fair end-to-end window-based congestion control," *IEEE/ACM Trans. Networking*, vol. 8, pp. 556-567, 2000.
- [32] J.R. Norris, *Markov Chains*, Cambridge University Press, Cambridge, 1997.

- [33] A.K. Parekh, and R.G. Gallager, "A generalized processor sharing approach to flow control in integrated services networks: the single node case," *IEEE/ACM Trans. on Networking*, vol. 1, no. 3, pp. 344-357, 1993.
- [34] Parekh, A.K. and R.G. Gallager, "A generalized processor sharing approach to flow control in integrated services networks: the multiple-node case," *IEEE INFOCOM 93*, San Francisco, 521-530, March 1993.
- [35] B.K. Ryu and A. Elwalid, "The importance of long-range dependence of VBR traffic engineering: myths and realities," *Proc. ACM SIGCOMM*, 3-14, August 1996.
- [36] Shwartz, A., and Weiss, A. *Large deviations for performance analysis, queues, communication and computing*, Chapman & Hall, London, 1995.
- [37] A. Simonian and J. Guibert, "Large deviations approximations for fluid queues fed by a large number of sources," *IEEE J. Selected Areas of Communications*, vol. 13, no. 6, 1081-1090.
- [38] R. Srikant, *The Mathematics of Internet Congestion Control*, Birkhäuser, Boston, 2003.
- [39] L. Tassiulas and A. Ephremides, "Stability properties of constrained queueing systems and scheduling for maximum throughput in multihop radio networks," *IEEE Trans. Automatic Control*, vol. 37, no. 12, December 1992.
- [40] L. Tassiulas and A. Ephremides, "Dynamic server allocation to parallel queues with randomly varying connectivity," *IEEE Trans. Information Theory*, vol. 39, March 1993.
- [41] L. Tassiulas, "Scheduling and performance limits of networks with constantly changing topology," *IEEE Trans. on Info. Theory*, Vol. 43, No. 3, pp. 1067-1073, May 1997.
- [42] R. L. Tweedie, The existence of moments of stationary Markov chains, *J. Appl. Prob.*, vol. 20, pp. 191 - 196, 1983.
- [43] W. Willinger, M.S. Taqqu, and A. Erramilli, "A bibliographical guide to self-similar traffic and performance modeling for modern high-speed networks." In *Stochastic Networks Theory and Applications*, F.P. Kelly, S. Zachary and I. Ziedins, Eds., Oxford Science, pp. 339-366, 1996.