Revenue and Stability of a Mechanism for Efficient Allocation of a Divisible Good

Sichao Yang and Bruce Hajek
Department of Electrical and Computer Engineering and the Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
email: syang8@uiuc.edu b-hajek@uiuc.edu

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Abstract A class of efficient mechanisms for allocating a divisible good is studied, which is more general than simple mechanisms discovered independently by Maheswaran and Başar and by us. Strategic buyers play a game by submitting one-dimensional bids, or signals, to the seller. The seller allocates the good in proportion to the bids and charges the buyers nonuniform prices according to the mechanism. Under some mild conditions on the valuation functions of the buyers, there is a unique Nash equilibrium point (NEP) and the allocation at the NEP is efficient. The prices charged to the buyers at the NEP are bounded above by, and can be made arbitrarily close to, the uniform market clearing price for price-taking buyers. The efficient NEP is globally stable. The work is motivated by the problem of rate allocation on the links of a communication network.

1 Introduction

This paper focuses on the problem of allocating a fixed quantity of a perfectly divisible good to a finite number of potential buyers. It is assumed that the bids, also known as signals or messages, issued by the buyers, are one-dimensional. A mechanism maps the vector of bids into a quantity of good and a payment for each buyer. Each buyer has a valuation function determining value as a function of the quantity of good received. The buyers engage in a game in which the payoffs are determined by the mechanism and the valuation functions of the buyers, and Nash equilibrium points (NEPs) are taken to be predictors of the behavior of the buyers. We are interested in allocations that are *efficient*, in the sense that they maximize the sum of the valuation functions of the buyers. In other words, we are considering Nash-implementation of efficient allocations, using one-dimensional bids.

Examples of allocation of a divisible good are the auction of an issue of government treasury notes [1], contracts for electrical power generation [5], and leasing shares of a tract of ocean for offshore oil exploration [28]. Our interest in allocation of a divisible good is motivated by the problem of allocating communication rate of a communication network, such as the Internet. Many of the links of a communication network can support a large communication rate, which can be as high as 10^{10} bits per second. The links can be shared by many flows, ranging from a handful of flows, to many thousands of flows. The seminal work of Kelly and his co-workers (see [4, 14]) sparked much interest in using pricing to study rate allocation in the Internet.

In most studies of congestion control in the Internet, the buyers are assumed to be price takers, and the network allocates transmission rates to the buyers in response to the bids. Kelly's work establishes that efficient (or socially optimal) allocation of transmission rates to the buyers can be obtained through a combination of price-taking behavior on the part of the buyers and proportionally fair allocation by the network. The overall optimality of the resulting equilibrium can be viewed as an instance of the first fundamental welfare theorem of economics [18]. A decentralized algorithm is given in [14] which allows the buyers to adjust their bids based on price feedback, yielding convergence to the socially optimal rate allocation. The assumption that the buyers are price takers is based on the fact that the computer programs that enter into the congestion game are embedded in computer operating systems and are not routinely tampered with by the vast majority of network users. The algorithms of [14] match well the currently most widely used transmission protocol on the Internet, the Transmission Control Protocol (see [25] for a survey of this and related algorithms). Nevertheless, there is the possibility that network users will become more strategic, perhaps by downloading transmission software updates, or perhaps due to implementation of actual monetary charges for high quality routing. This has sparked work similar to the model of [4, 14], but with the buyers exhibiting strategic behavior.

The proportionally fair network allocation mechanism advocated by Kelly et al. [4, 14] is a method for simultaneously making rate allocations on all links of a network. The simplest case is a network with one link, such that the rate on the link is allocated in simple proportion to the bids, which are payments. The result is a uniform price[1] for the link, equal to the sum of the bids divided by the link capacity. A game among the buyers can thus be defined for this proportional allocation on a single link. This was first done in the context of communication networks in [8, 16] which show that, under mild conditions or relaxations, NEPs exist. A local stability result for the NEP is given by Maheswaran and Başar [16]. Subsequently, Johari and Tsitsiklis [13] showed that the worst efficiency ratio at NEPs is 75% for this game. They also showed that this result extends to general topologies, if each buyer submits a separate bid for each link that it will use. Hajek and Yang [9] show that an allocation based on sum bids as in [14], rather than itemized bids as in [13], can lead to nonexistence of NEPs. Furthermore, the sum of the valuations at the NEPs may not be the maximum value, i.e. the NEPs are not efficient, in both the itemized [13] and sum bid [9] games.

This paper is concerned with Nash implementation of efficient allocation. The concept of Nash equilibrium has a well defined mathematical meaning, but to fully justify its use as an implemen-

[1] In this paper, "price" always refers to payment per unit quantity.

tation mechanism there needs to be a way for the NEP to be found. The buyers themselves can compute the NEP under the assumption of *complete information*, whereby each buyer knows the preferences of all buyers. It's best if the NEP is unique, but if not, the buyers would have to use a rule to compute the same NEP. Nash implementation in the economics literature is typically based on the assumption of complete information – see E. Maskin and Sjöström [19] for a thorough discussion on this point. Complete information might be a reasonable assumption in a scenario such as power industry auctions, in which buyers can open their facilities for mutual inspection. Or, gasoline rationing among a group of motorists, in which the motorists could open their fuel tanks for mutual inspection. Or, sharing of a kill, by a group of hungry lions, who can assess each others' hunger by inspecting each others' bellies. Once the buyers have complete information, each buyer can compute its allocation under the NEP. On one hand, if each buyer can unilaterally take her allocation, or if a central broker makes the allocations and the central broker also has complete information, then no communication is needed. On the other hand, if there is a central broker without complete information, then the buyers may need to communicate to the broker. The g mechanism proposed in this paper can be used to implement efficiency under Nash equilibrium with complete information.

In the context of communication networks, in which the population of buyers is in constant flux, and different buyers compete for bandwidth on different links, the complete information assumption is difficult, if not impossible, to justify. An alternative justification of Nash implementation is put forth in Section 4. It is based on stability of the NEP and *local information*, which we feel is suitable in the context of rate allocation in high speed networks.

Maheswaran and Başar [17] introduced a family of mechanisms, which we call simple g-mechanisms, for implementing efficient allocation of a divisible good, using one-dimensional bids, and they established existence, uniqueness, and efficiency of NEPs.[2] In this paper, we present a more general family of mechanisms, which we call g-mechanisms, for implementing efficient allocation of a divisible good, using one-dimensional bids. The bids are not payments, and the prices are not uniform. It is shown under mild regularity assumptions or relaxations, that an NEP exists, is unique, and is efficient.

There are four further contributions of this paper. First, the prices paid and revenue generated for the seller are investigated. It is shown that the prices paid for the good for any (under regularity restrictions) efficient allocation mechanism are less than the market clearing price for price-taking buyers (Section 2.3), and that g-mechanisms with an appropriate choice of the function g can yield revenue arbitrarily close to the upper bound (Section 3.3). Second, it is shown that, under a set of reasonable assumptions, all mechanisms for Nash implementation of efficient allocation using allocation proportional to bids are g-mechanisms (Section 3.2). Third, a globally stable decentralized algorithm is given under which buyers arrive at an NEP for a simple g-mechanism

[2]The simple g-mechanisms and associated existence, uniqueness, and efficiency of NEPs was independently discovered by Yang and Hajek [29] with slightly different details.

(Section 4). Fourth, a comparison is given to mechanisms inspired by the literature on Nash implementation theory with minimum dimension message spaces (Section 5).

Section 2 briefly reviews two allocation mechanisms: proportional allocation with bids equal to payments, for either price taking buyers or strategic buyers, and the VCG mechanism [2, 6, 26], and it concludes with the general upper bound on revenue. Section 3 presents the family of g-mechanisms and their properties. Conclusions and questions for future investigation are given in Section 6.

2 Market model

Suppose there are N buyers in the market, with $N \geq 2$. Let the valuation function [3] of buyer i be $U_i(x_i)$, where x_i is the quantity of the good allocated by the seller to buyer i. In this paper, U_i is assumed to be a strictly concave, strictly increasing and continuously differentiable function on $[0, +\infty)$. It is permitted that $U_i'(0) = +\infty$. The total quantity of the good that the seller has to sell is $C > 0$. We say that C is the *capacity constraint*. As in [14], the system problem is defined as,

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{N} U_i(x_i) \qquad (1) \\ \text{subject to:} \quad & \sum_{i=1}^{N} x_i \leq C \text{ and } x_i \geq 0, \ \forall\ i. \end{aligned}$$

A solution to the system problem maximizes the aggregate value of all the buyers within the capacity constraints. Thus, an allocation vector $\boldsymbol{x}$ is efficient if and only if it is the solution to the system problem. The objective function of the system problem is strictly concave and the feasible set is convex and compact. Furthermore, all the feasible vectors are regular. Therefore, a unique efficient allocation exists and is characterized by the Kuhn-Tucker condition: There exists a $\lambda \geq 0$ such that

$$\begin{cases} U_i'(x_i) = \lambda, & \text{if } x_i > 0 \\ U_i'(0) \leq \lambda, & \text{if } x_i = 0 \end{cases} \qquad (2)$$
$$\sum_{i=1}^{N} x_i = C$$

Furthermore, λ is unique since U_i is continuously differentiable for all i. For the reasons explained in Section 2.1, λ is called the *market clearing price for price taking buyers.*

We consider auction mechanisms that can be formulated as a game. There are $N \geq 2$ buyers competing for a finite quantity C of divisible good. A *mechanism* consists of a triple $(\mathcal{B}, \boldsymbol{x}, \boldsymbol{m})$,

[3] Buyer's utility function is given by the sum of valuation function and the payment, the so-called quasilinear utility function.

where $\mathcal{B}$ is the set of allowed bid vectors of the form $\boldsymbol{b} = (b_1, \ldots, b_N)$, $\boldsymbol{x}$ is the allocation rule, and $\boldsymbol{m}$ is the payment rule. Each buyer i submits a bid b_i to the seller. Given a bid vector $\boldsymbol{b} = (b_1, ..., b_N)$, let $\boldsymbol{b}_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_N)$. We use B and B_{-i} to denote the sums $B = \sum_j b_j$ and $B_{-i} = \sum_{j \neq i} b_j$. The seller allocates quantity $x_i(b_i; \boldsymbol{b}_{-i})$ to buyer i for a charge $m_i(b_i; \boldsymbol{b}_{-i})$. The mechanism $(\mathcal{B}, \boldsymbol{x}, \boldsymbol{m})$ and valuation functions $U_1, ..., U_N$ determine the following game:

GAME:

Buyer i has bid b_i and payoff function $\Pi_i(b_i; \boldsymbol{b}_{-i})$, which is quasilinear in the payment, given by $\Pi_i(b_i; \boldsymbol{b}_{-i}) = U_i(x_i(b_i; \boldsymbol{b}_{-i})) - m_i(b_i; \boldsymbol{b}_{-i})$. Each buyer tries to maximize her payoff unilaterally by adjusting her own bid. A bid vector $\boldsymbol{b}$ is, by definition, an NEP, if for all i:

$$\Pi_i(b_i; \boldsymbol{b}_{-i}) \geq \Pi_i(\bar{b}_i; \boldsymbol{b}_{-i}), \ \forall \ \bar{b}_i \text{ s.t. } (\bar{b}_i; \boldsymbol{b}_{-i}) \in \mathcal{B}. \tag{3}$$

An NEP $\boldsymbol{b}$ is *efficient* if the allocation $\boldsymbol{x}(\boldsymbol{b})$ is efficient.

Two mechanisms and a general upper bound on revenue are discussed in the remainder of this section, to set the context for the g-mechanisms presented in Section 3.

2.1 Proportional allocation with payments as bids

Under the mechanism of proportional allocation with payments as bids, each bid b_i is a nonnegative real number, so that $\mathcal{B} = \mathbb{R}_+^N$. [4] The good is allocated in proportion to the bids and the payment of each buyer equals her bid. The mechanism is written in detail as follows:

$$\text{Allocation rule:} \qquad x_i = \begin{cases} \dfrac{b_i}{B} C, & \text{if } \boldsymbol{b} \neq 0, \\ 0, & \text{if } \boldsymbol{b} = 0, \end{cases} \tag{4}$$
$$\text{Payment rule:} \qquad m_i = b_i.$$

The buyers are charged a uniform price $p = \frac{B}{C}$. (In this paper, the word "price" means price per unit.) The behavior of this mechanism is discussed next, first for price taking buyers and then for strategic buyers.

A price taking buyer as in [14] acts as if the price is unchangeable. That is, the seller is a leader and the buyers are followers. If a price p with $p > 0$ is announced, the response of a price taking buyer with valuation function U_i is the bid b_i, or equivalently the quantity $x_i = \frac{b_i}{p}$, which maximizes $\Pi(b_i) = U_i(\frac{b_i}{p}) - b_i$. The response is determined by

$$\begin{cases} U_i'\left(\dfrac{b_i}{p}\right) = p, & \text{if } b_i > 0 \\ U_i'(0) \leq p, & \text{if } b_i = 0 \end{cases} \tag{5}$$

[4] In the paper, we use $\mathbb{R}_+$ for the set of nonnegative real numbers and $\mathbb{R}_{++}$ for the set of positive real numbers.

In this light, the variable λ appearing in the condition (2) for efficient allocation has the following interpretation. It is the price such that the responses expressed as quantities (the x_i's) sum to the total capacity. That is why the value λ is called the market clearing price for price taking buyers. A similar interpretation is that the efficient allocation $\boldsymbol{x}$ and price λ are the equilibrium allocation vector and price for the proportional allocation mechanism with price taking buyers. Note that the revenue generated by the price taking buyers is λC. It is shown in Section 2.3 that, in a certain sense, if an efficient mechanism is applied with strategic buyers, the revenue is less than λC. It is shown in Section 3.3 that simple g-mechanisms can yield revenue arbitrarily close to λC.

A strategic buyer takes into account her own influence on the market price. The payoff of buyer i is

$$\Pi_i(b_i; \boldsymbol{b}_{-i}) = U_i\left(\frac{b_i}{B}C\right) - b_i \tag{6}$$

This determines a game in which each buyer tries to maximize her payoff by adjusting her bid. There is a unique NEP $\boldsymbol{b}$ which, together with a price p, is the solution to the following [8, 16].

$$\begin{aligned} & U_i'(x_i)\left(1 - \frac{x_i}{C}\right) = p \\ & \text{where } p = \frac{B}{C} \text{ and } x_i = \frac{b_i}{p}, \ \forall\ i \end{aligned} \tag{7}$$

The price p is the uniform price that the seller charges in this mechanism, and $p < \lambda$.

2.2 VCG mechanism

In contrast to the mechanisms using one-dimensional bids, the Vickrey-Clark-Groves (VCG) mechanism [2, 6, 26] requires each buyer to report her whole valuation function to the seller. Let $b_i(x)$ be the reported valuation function of buyer i. Clearly $\mathcal{B}$ is an infinite dimensional space for the VCG mechanism. Let

$$W_{-i}(x) = \max_{\boldsymbol{x}_{-i}: \sum_{j \neq i} x_j \leq C - x} \sum_{j \neq i} b_j(x_j) \tag{8}$$

The allocation and payment rule of the VCG mechanism are the following:

$$\text{Allocation rule: } \boldsymbol{x} \in \operatorname*{argmax}_{\sum_j x_j \leq C} \sum_j b_j(x_j) \tag{9}$$

$$\text{Payment rule: } \quad m_i = W_{-i}(0) - W_{-i}(x_i) \tag{10}$$

Hence, the payoff of buyer i is

$$\Pi_i(x_i) = (U_i(x_i) + W_{-i}(x_i)) - W_{-i}(0) \tag{11}$$

In (11), $W_{-i}(0)$ does not depend on x_i. If buyer i reports truthfully, i.e. $b_i(x) = U_i(x)$, then the seller seeks a choice of x_i that maximizes the sum of the first two terms of payoff. Hence, truth reporting is an optimal strategy, no matter what the other buyers report. All the buyers reporting their true valuations is an efficient NEP for the VCG mechanism.

In the payment rule of the VCG mechanism, $W_{-i}(0)$ is the maximum social welfare if buyer i is excluded and $W_{-i}(x_i)$ is the maximum welfare of other buyers if quantity x_i is allocated to buyer i. The key to the efficiency of the VCG mechanism is that the payment of a buyer compensates precisely for the loss of other buyers' welfare because of her competition.

2.3 Price bounds of efficient mechanisms

A buyer is characterized by her valuation function, and is said to be a *regular buyer* if the valuation function is strictly increasing, concave, and differentiable with a piecewise continuous second derivative bounded away from zero. Given an integer $N \geq 1$, a set of N buyers can be identified with a vector of N valuation functions, $(U_1, \ldots, U_N)$. The following proposition is proved in Appendix A.

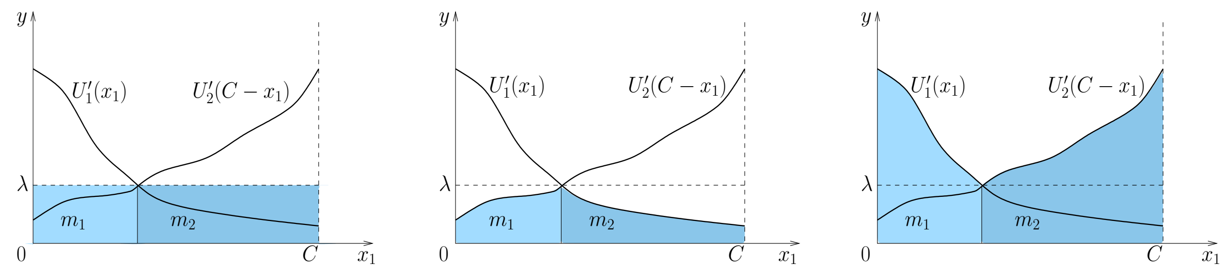
Proposition 2.1 (Price bounds of efficient mechanisms) *Fix $N \geq 1$ and suppose $(\mathcal{B}, \boldsymbol{x}, \boldsymbol{m})$ is a mechanism for the allocation of a divisible good among N buyers. (The bids need not be one dimensional). Suppose the following two conditions hold:*

1. *An efficient NEP exists for any profile of regular valuation functions for N buyers.*
2. *For each buyer i, there exists a bid b_i^o such that $x_i(b_i^o; \boldsymbol{b}_{-i}) = 0$ and $m_i(b_i^o; \boldsymbol{b}_{-i}) = 0$ for all $\boldsymbol{b}_{-i}$.*

Then, for any profile of regular valuation functions for N buyers and $\delta > 0$, there is at least one efficient NEP at which the price p_i^* charged to buyer i is less than $\lambda + \delta$ for all i. (Here, λ is the market clearing price for price taking buyers, discussed in Section 2.1.)

In particular, the revenue generated by the VCG mechanism is less than λC. This result on VCG revenue is given in a combinatorial context in [7].

Three examples for the revenues of different mechanisms and two buyers are shown in Figure 1. In uniform pricing for price taking buyers (Figure 1(a)), the prices at the efficient allocation equal λ. For the VCG mechanism (Figure 1(b)), the prices are less than λ, which meets the conclusion of Proposition 2.1, since the VCG mechanism can derive an efficient allocation for any profile of regular valuation functions. However, if the seller has the full information of buyers' valuation functions, an aggressive discriminatory pricing mechanism (Figure 1(c)) can be applied in which the buyer's payment equals the integration of her valuation of the allocation. The prices exceed λ at the equilibrium point. However, this mechanism does not yield an efficient allocation for other sets of buyers. This example shows that, in Proposition 2.1, the condition that the efficient NEPs exist for *any profile of regular valuation functions* is important. If a mechanism is designed for a specific profile of valuation functions, then the price may greatly exceed λ for all NEPs.



(a) Uniform pricing for price taking buyers

(b) VCG mechanism

(c) Aggressive discriminatory

Figure 1: Comparison of revenues for three different mechanisms.

The price upper bound in Proposition 2.1 applies to at least one efficient NEP. Moreover, an example in Appendix A shows that, for a mechanism that satisfies the conditions in Proposition 2.1, there is an efficient NEP at which the prices of the buyers exceed λ, although there is another NEP at which the prices are less than λ.

3 A class of efficient pricing mechanisms

A drawback of the VCG mechanism is that the bids are functions. We present efficient pricing mechanisms which are somewhat similar to the VCG mechanism, but which use one-dimensional bids, i.e. $\mathcal{B} = \{\boldsymbol{b} : b_i \geq 0, \ \forall\ i\}$. The allocation and payment rules of such a mechanism are given by:

$$\text{Allocation rule: } x_i(\boldsymbol{b}) = \begin{cases} \dfrac{b_i}{B}C, & \text{if } \boldsymbol{b} \neq \boldsymbol{0} \\ 0, & \text{if } \boldsymbol{b} = \boldsymbol{0} \end{cases} \tag{12}$$

$$\text{Payment rule: } \quad m_i(\boldsymbol{b}) = \begin{cases} CB_{-i}\displaystyle\int_0^{b_i} \frac{g(t;\boldsymbol{b}_{-i})}{(t+B_{-i})^2}dt, & \text{if } B_{-i} \neq 0 \\ 0, & \text{if } B_{-i} = 0 \end{cases} \tag{13}$$

where $g(\boldsymbol{b}) : \mathbb{R}_+^N \to \mathbb{R}_+$ is a continuous, nondecreasing function such that $g(\theta\boldsymbol{b})$ is a strictly increasing map from $\mathbb{R}_+$ onto $\mathbb{R}_+$ whenever $\boldsymbol{b} \neq 0$. It is not difficult to verify the following regularity properties: (i) $\boldsymbol{x}$ is continuous on $\mathbb{R}_+^N - \{0\}$, (ii) m_i is continuous on $\mathbb{R}_+^N$, (iii) $m_i(\boldsymbol{b}) = 0$ if $b_i = 0$ or $B_{-i} = 0$, (iv) $\frac{\partial x_i}{\partial b_i}$ exists and is continuous on $\mathbb{R}_+^n$, and (v) $\frac{\partial m_i}{\partial b_i}$ exists and is continuous on $\mathbb{R}_+^n - \{\boldsymbol{0}\}$. Furthermore,

$$\frac{\partial x_i}{\partial b_i} = \frac{CB_{-i}}{B^2} \text{ on } \mathbb{R}_+^N \quad \text{and} \quad \frac{\partial m_i}{\partial b_i} = \frac{CB_{-i}}{B^2}g(\boldsymbol{b}) \text{ on } \boldsymbol{b} \neq 0 \tag{14}$$

The payoff of buyer i is

$$\Pi_i(b_i;\boldsymbol{b}_{-i}) = U_i\left(\frac{b_i}{B}C\right) - CB_{-i}\int_0^{b_i} \frac{g(t;\boldsymbol{b}_{-i})}{(t+B_{-i})^2}dt, \quad \text{if } B_{-i} \neq 0 \tag{15}$$

We call $g(\boldsymbol{b})$ the *marginal price function* since $g(\boldsymbol{b}) = \frac{\partial m_i/\partial b_i}{\partial x_i/\partial b_i}$, and we call the mechanisms specified by (12) and (13) *g-mechanisms* since they are determined by the choice of g.

A special case is that $g(\boldsymbol{b})$ depends on $\boldsymbol{b}$ only through the sum of bids, B. By a slight abuse of notation in that case, we write $g(B)$ instead of $g(\boldsymbol{b})$, and then $g(u)$ is a continuous and strictly increasing onto map from $\mathbb{R}_+$ to $\mathbb{R}_+$. In this case the payment has a simple form

$$m_i(\boldsymbol{b}) = B_{-i}[\varphi(B) - \varphi(B_{-i})] \tag{16}$$

where φ is a function such that $\varphi'(u) = \frac{g(u)C}{u^2}$. We call the mechanisms specified by (12) and (16) *simple g-mechanisms*. Examples of simple g-mechanisms are given by $g(u) = \frac{u^\alpha}{C}$ for $\alpha > 0$, corresponding to

$$\varphi(u) = \begin{cases} \log u, & \text{if } \alpha = 1. \\ \dfrac{u^{\alpha-1}}{\alpha - 1}, & \text{if } \alpha > 0 \text{ and } \alpha \neq 1. \end{cases} \tag{17}$$

This yields particularly simple forms of payment functions:

$$\begin{aligned} m_i(\boldsymbol{b}) &= B_{-i} \log(1 + \tfrac{b_i}{B_{-i}}) && (\text{for } \alpha = 1) \\ m_i(\boldsymbol{b}) &= b_i B_{-i} && (\text{for } \alpha = 2) \end{aligned}$$

The family of simple g-mechanisms was independently proposed by Maheswaran and Başar [17]. They called $g(u)$ the generator function and established existence, uniqueness, and efficiency of the NEPs. Our initial discovery of simple g-mechanisms started with the representation (16) for a function φ such that $g(u) = \frac{u^2\varphi'(u)}{C}$ is a strictly increasing map of $[0, \infty)$ onto $[0, \infty)$. However, the form of payment (16) does not generalize to the case of general g-mechanisms. The class of all g-mechanisms with certain natural properties is equal to the set of all mechanisms for Nash implementation of efficient allocation based on allocation proportional to one-dimensional bids (signals), as shown in Section 3.2.

Maheswaran and Başar [17] proved uniqueness of NEPs under slightly different assumptions from ours. Under the proportional allocation defined in (12), a buyer's allocation is a discontinuous function of her bid if $B_{-i} = 0$. Maheswaran and Başar address this problem by assuming that the seller has a reserved bid $\epsilon > 0$. Under this assumption, $B_{-i} \geq \epsilon$, and they prove that there is a unique NEP and the NEP is efficient in the limit as ϵ approaches 0. In contrast, we derive uniqueness and efficiency of the NEP by introducing an assumption on the buyers' valuation functions (Assumption 3.1). In case the assumption does not hold, we characterize the set of NEPs and give an interpretation based on bluffing behavior, and give a relaxation, different from the one of [17], with a unique NEP. Also, the globally stable algorithm we propose converges to an efficient NEP even if the assumption does not hold.

3.1 Existence and uniqueness of NEP

The following assumption will sometimes be invoked. Section 3.4 discusses the mechanism in case the assumption fails.

Assumption 3.1 $U_i'(0) = +\infty$ *for at least two values of* i*.*

The following proposition shows the existence, uniqueness, and efficiency of the NEP when a g-mechanism is applied under Assumption 3.1 and it is proved in Appendix B.

Proposition 3.1 *Fix a marginal price function* g *and suppose Assumption 3.1 holds. There is a unique NEP* $\boldsymbol{b}^*$*, and it is the unique solution to*

$$\begin{cases} U_i'(x_i) & = & g(\boldsymbol{b}), & \text{if } \ b_i > 0 \\ U_i'(0) & \leq & g(\boldsymbol{b}), & \text{if } \ b_i = 0 \end{cases}, \ \forall \ i \tag{18}$$

$$\text{where } x_i = \frac{b_i}{B} C.$$

The allocation at $\boldsymbol{b}^*$ *is efficient. At the NEP,* $g(\boldsymbol{b}^*) = \lambda$*, where* λ *is the market clearing price for price taking buyers. (Note that (18) implies* $b_i > 0$ *if* $U_i'(0) = +\infty$*.)*

Remark 3.1 *Suppose* i *is a buyer and* $\boldsymbol{b}$ *is a bid vector such that* $\boldsymbol{b}_{-i}$ *satisfies the sufficient condition for an NEP (for* b_i *fixed). Then if* b_i *is increased slightly, the first order increase in the payment of buyer* i *is equal to the first order decrease in the sum of values of the other buyers.*

Suppose buyer i *changes her bid from* b_i *to* $b_i + \delta$*. Then ignoring* $o(\delta)$ *terms, the change of the* j*-th buyer's valuation is*

$$\Delta_{U_j} = \frac{\partial [U_j(x_j(b_i; \boldsymbol{b}_{-i}))]}{\partial b_i} \delta = U_j'(x_j) \frac{\partial x_j}{\partial b_j} \delta = g(\boldsymbol{b}) \frac{\partial x_j}{\partial b_i} \delta,$$

and

$$\Delta_{m_i} \quad = \quad \frac{\partial [m_i(b_i; \boldsymbol{b}_{-i})]}{b_i} \delta = g(\boldsymbol{b}) \frac{\partial x_i}{\partial b_i} \delta$$

Since $\sum_j x_j = C$*,* $\Delta_{m_i} + \sum_{j \neq i} \Delta_{U_j} = 0$*.*

Remark 3.1 shows that, up to order $o(\delta)$, the payment of any buyer is equal to the externality she exerts on the other buyers at the equilibrium. The same principle is behind VCG mechanism.

3.2 On the generality of g-mechanisms

In this section we show that g-mechanisms, under reasonable assumptions, are the only mechanisms yielding Nash implementation of efficient allocation for allocation proportional to bids. Consider the following assumptions on a mechanism $(\mathcal{B}, \boldsymbol{x}, \boldsymbol{m})$.

Assumption G1: There are at least two buyers (i.e. $N \geq 2$), the set of allowable bid vectors $\mathcal{B}$ is equal to $\mathbb{R}_+^N$, and allocation is proportional to bids: $x_i(\boldsymbol{b}) = b_i C/B$ if $\boldsymbol{b} \neq \mathbf{0}$, and $x_i(\boldsymbol{b}) = 0$ if $\boldsymbol{b} = \mathbf{0}$.

Assumption G2: The function m_i is continuous on $\mathbb{R}_+^N$ and is zero on $\mathbb{R}_+^N \cap \{b_i = 0\}$, for $1 \leq i \leq N$. Furthermore, on $\mathbb{R}_+^N \cup \{B_{-i} > 0\}$, the partial derivative $\frac{\partial m_i}{\partial b_i}$ exists, is continuous, and is strictly positive, for $1 \leq i \leq N$.

Assumption G3: For any set of valuation functions $U_1, \ldots, U_N$, such that each U_i is twice continuously differentiable, strictly concave, and $U_i'(0+) = +\infty$, there is a unique NEP, and it is efficient.[5]

For $1 \leq i \leq N$, define the function g_i with domain $\mathbb{R}_+^N \cap \{B_{-i} > 0\}$ by

$$g_i = \frac{\frac{\partial m_i}{\partial b_i}}{\frac{\partial x_i}{\partial b_i}} = \frac{B^2}{B_{-i}C}\frac{\partial m_i}{\partial b_i}. \tag{19}$$

Under Assumption G2, g_i is continuous on its domain. Then $g_i(\boldsymbol{b})$ is the marginal cost for buyer i, if at least one other bid is positive. Thus, if bid vector $\boldsymbol{b} \in R_{++}^N$ is an NEP for some set of valuation functions $U_1, \ldots, U_N$, then

$$U_i'(x_i(\boldsymbol{b})) = g_i(\boldsymbol{b}) \text{ for } 1 \leq i \leq N. \tag{20}$$

The following assumption is a reasonable way to insure that the condition (20) is also sufficient for $\boldsymbol{b}$ to be an NEP:

Assumption G4: For any i with $1 \leq i \leq N$ and any $\boldsymbol{b}_{-i}$ fixed, $g_i(\boldsymbol{b})$ (also written $g_i(b_i, \boldsymbol{b}_{-i})$) is a nondecreasing function of b_i.

Proposition 3.2 *Suppose $(\mathcal{B}, \boldsymbol{x}, \boldsymbol{m})$ is a mechanism satisfying Assumptions G1-G4. Then it is a g-mechanism for some marginal price function g.*

The proposition above is proved in Appendix C.

3.3 Buyers' payments and seller's revenue

Next, we will discuss the relationship of the buyers' price and sellers' revenue to the choice of g-functions. For any buyer i, the *average price*, or simply price for short, is defined as her payment per unit rate, i.e. $p_i = \frac{m_i}{x_i}$ given that $x_i > 0$. The revenue of the seller is the sum of the payment of all the buyers, i.e $R = \sum_i m_i$.

The following proposition complements Proposition 2.1. Regarding the upper bound on prices, it shows that the revenue for the g-mechanism and any choice of g is strictly smaller than λC. The main point of the following proposition is that the revenue can be made arbitrarily close to λC, or arbitrarily close to zero, by using g from a particular one parameter class of functions. The proofs of the propositions of this section are given in Appendix D.

[5] Assumption G3 as stated is weaker than if it were modified to require the unique, efficient NEP whenever $U_i'(0) = +\infty$ for at least two buyers i.

Proposition 3.3 (Prices for g-mechanisms) *Suppose Assumption 3.1 holds. (i) For a g-mechanism, the price p_i^* charged to any buyer i at the NEP satisfies $0 \le p_i^* < \lambda$ for all i. (ii) For the simple g-mechanism with marginal price function $g_\alpha(u) = \frac{u^\alpha}{C}$ for $\alpha > 0$, the prices at the NEP satisfy $\lim_{\alpha\to 0} p_i^* = \lambda$ and $\lim_{\alpha\to\infty} p_i^* = 0$ for all i.*

Notice that the prices p_i's are continuous functions of α. An immediate corollary of the proposition is the following:

Corollary 3.1 *Suppose Assumption 3.1 holds. (i) For g-mechanisms, the supremum of the seller's revenue R with respect to the choice of g is λC and the infimum of the seller's revenue R with respect to the choice of g is 0. (ii) For any value of $R \in (0, \lambda C)$, there exists an $\alpha > 0$ so that the revenue of the seller is R, for the simple g-mechanism with marginal price function $g_\alpha(u) = \frac{u^\alpha}{C}$.*

Remark 3.2 *Corollary 3.1 shows that, if the cost of resource provision is between 0 and λC, there is a g-mechanism that implements the allocation maximizing the total valuation and is budget balanced.*

It is interesting to compare our result in Proposition 3.3 to the revenue results in the auction theory in economics literature. A well-known fact from mechanism design with incomplete information states that the allocation rule uniquely determines the prices, which is called revenue equivalence principle [20]. In Bayesian Nash equilibria, the expected revenues of the seller are the same for all mechanisms with identical allocation. However, Proposition 3.3 shows that, in incomplete information setting, the seller's revenue can be different for mechanisms implementing the same allocation in Nash equilibria. In [7] by Gul and Stacchetti, the authors show that if the market is replicated enough times, the prices in the VCG mechanism and in the Walrasian equilibrium are the same. This result is intuitively true because a large amount of replication of a small market will form a competitive market. The prices in g-mechanisms have the same upper bound. However, the upper bound can be approached by adjusting the mechanism instead of replicating the market. Therefore, in a small market which is not competitive, the price in Walrasian equilibrium is still achievable by g-mechanisms asymptotically.

Additional properties of g-mechanisms and simple g-mechanisms are given in the remainder of this section.

Proposition 3.4 (Monotonicity of prices) *For a g-mechanism, the price p_i is an increasing function of $\boldsymbol{b}$.*

Proposition 3.5 (Volume Discounts) *For a simple g-mechanism, the prices charged to the buyers are in the reverse order of the buyers' bids and allocations (i.e. if $b_i < b_j$, then $x_i < x_j$ and $p_i > p_j$). This property does not hold for g-mechanisms in general.*

Proposition 3.6 (Monotonicity of equilibrium prices) *For a simple g-mechanism under Assumption 3.1, if a new buyer U_{N+1} joins the auction consisting of buyers $(U_1, \ldots, U_N)$, the price p_i^* at the NEP increases for $1 \le i \le N$. This property does not hold for g-mechanisms in general.*

The following examples show that Proposition 3.5 and Proposition 3.6 don't necessary hold for general g-mechanisms.

Example 3.1 (A counter example for volume discounts) *Let $g(\boldsymbol{b}) = \sum_{i=1}^N h(b_i)$, where $h(t) = \min\{t, 1\} + \epsilon t$. Then*

$$\begin{aligned} p_i(\boldsymbol{b}) &= \frac{B_{-i}B}{b_i} \int_0^{b_i} \frac{h(t) + \sum_{j \neq i} h(b_j)}{(t + B_{-i})^2} dt \\ &= \frac{B_{-i}B}{b_i} \int_0^{b_i} \frac{h(t)}{(t + B_{-i})^2} + \sum_{j \neq i} h(b_j) \end{aligned} \tag{21}$$

Consider a bid vector $\boldsymbol{b} = (2, 1)$. We derive $p_1(\boldsymbol{b}) = \frac{3}{2}\log 2 + \frac{1}{2} + O(\epsilon) = 1.539 + O(\epsilon)$ and $p_2 = 6 \log \frac{3}{2} - 1 + O(\epsilon) = 1.432 + O(\epsilon)$. In the example, $p_1(\boldsymbol{b}) > p_2(\boldsymbol{b})$ while $b_1 > b_2$.

Example 3.2 (A counter example for the price increase in NEP as number of buyers increase) *We fix the marginal price function $g(\boldsymbol{b})$ to be the same as in Example 3.1. Suppose $\boldsymbol{b} = (2, 2)$ is an NEP for a population of 2 buyers. It is possible that a 3^{rd} buyer is added to the population and the new NEP is $\tilde{\boldsymbol{b}} = (1, 1, 3\epsilon)$, because $g(\tilde{\boldsymbol{b}}) = 2 + 5\epsilon > 2 + 4\epsilon = g(\boldsymbol{b})$. From (21), consider the price of buyer 1, $p_1(\boldsymbol{b}) = 4 \log \frac{3}{2} + O(\epsilon) = 1.622 + O(\epsilon)$ and $p_1(\tilde{\boldsymbol{b}}) \doteq 2 \log 2 + O(\epsilon) = 1.386 + O(\epsilon)$. In the example, the price in NEP decreases as a new buyer is added.*

3.4 What if Assumption 3.1 doesn't hold

Propositions 3.1 and 3.3 are based on Assumption 3.1, stating that at least two buyers i have $U_i'(0) = +\infty$. This section explores the game in case Assumption 3.1 doesn't hold. The proofs of the propositions of this section are given in Appendix E.

Definition 3.2 *Buyer i is a* dominant *buyer in the market if $U_i'(C) \ge U_j'(0)$ for all $j \neq i$.*

Remark 3.3 *Buyer i gets all of the good under the efficient allocation if and only if she is a dominant buyer. (This can be shown directly from (2)).*

Proposition 3.7 *Suppose the g-mechanism is applied in the auction GAME. The set of conditions (18) has a unique solution $\boldsymbol{b}^*$ and it is an efficient NEP. If Assumption 3.1 holds, there are no other NEPs. If Assumption 3.1 fails to hold, there exist infinitely many NEPs. Specifically, $\hat{\mathcal{B}} \cup \{\boldsymbol{b}^*\}$ is the set of all NEPs, where $\hat{\mathcal{B}} = \cup_i \hat{\mathcal{B}}_i$ and $\hat{\mathcal{B}}_i = \{\boldsymbol{b} : b_j = 0$ and $g(b_i) \ge U_j'(0)$ for all $j \neq i\}$. Furthermore, if there exists a dominant buyer $\bar{i}$, then $\boldsymbol{b}^* \in \hat{\mathcal{B}}_{\bar{i}} \subset \hat{\mathcal{B}}$ and $\hat{\mathcal{B}}_{\bar{i}}$ is the set of all efficient NEPs. If there doesn't exist a dominant buyer, then $\boldsymbol{b}^* \notin \hat{\mathcal{B}}$ and none of the NEPs in $\hat{\mathcal{B}}$ are efficient.*

Remark 3.4 *When* $\boldsymbol{b} \in \hat{\mathcal{B}}_i$*, the bid of buyer* i *can be seen as a bluff in the game. At* $\boldsymbol{b}$*, buyer* i *bluffs the other buyers by claiming a high marginal price* $g(b_i; \mathbf{0}_{-i})$ *that exceeds the other buyers' largest marginal valuation. The other buyers are bluffed and are not willing to take part in the bidding competition, which leaves the payment of buyer* i *zero. A buyer with* $U_i'(0) = +\infty$ *is unbluffable. So if Assumption 3.1 holds, it is equivalent to say that there are at least two unbluffable buyers.*

In next part of this subsection, we introduce an ϵ*-extended GAME* under which the Assumption 3.1 always holds.

ϵ*-extended GAME:*

Keep all the other settings the same as *GAME* except assume that there are two additional buyers, called *virtual buyers*, in the game, which are denoted as buyer I and buyer $I\!I$. The virtual buyers have valuation functions $U_I(x_I) = w\epsilon \log x_I$ and $U_{I\!I}(x_{I\!I}) = (1-w)\epsilon \log x_{I\!I}$ respectively for some $0 < w < 1$. They are also strategic buyers who select their bids $b_I \geq 0$ and $b_{I\!I} \geq 0$ to maximize their own profits. Let $\tilde{\boldsymbol{b}}$ be the extension of $\boldsymbol{b}$ to included the bids of the virtual buyers: $\tilde{\boldsymbol{b}} = (b_1, ..., b_N, b_I, b_{I\!I})$ and similarly let $\tilde{\boldsymbol{x}} = (x_1, ..., x_N, x_I, x_{I\!I})$. From Proposition 3.1, there is a unique NEP $\tilde{\boldsymbol{b}}_\epsilon$ in the ϵ*-extended GAME.*

Proposition 3.8 *The limit* $\tilde{\boldsymbol{b}} = \lim_{\epsilon \to 0} \tilde{\boldsymbol{b}}_\epsilon$ *exists. Let* $\boldsymbol{b}$ *be the vector composed of the first N elements of* $\tilde{\boldsymbol{b}}$*. Then* $\boldsymbol{b}$ *is the efficient NEP of* GAME*.*

4 Global stability of the NEP under a simple g-mechanism

In the previous section, a mechanism for which the NEP is efficient is presented. In this section, we investigate the stability of the NEP. We propose an algorithm, which requires the buyers to update their bids based on *local information*, to reach the NEP in simple g-mechanisms. The information a buyer requires is local in two senses: (1) only the effect at small changes is needed and (2) knowledge of the valuation functions and allocations of other buyers, or even the number of other buyers, is not needed. The principle of the algorithm can be described as follows. All the buyers start with non-zero bids and continuously update their bids. The update rate of each buyer is proportional to her current bid, and the update direction is determined by comparing the buyer's marginal valuation to the marginal market price. In the simple g-mechanism, buyer i can determine the marginal price $g(B)$ from her own bid and allocation by $g(B) = g(\frac{b_i C}{x_i})$, while this is not generally true for g-mechanism. If the marginal valuation is larger than the marginal market price, the buyer increases her bid. If the marginal valuation is smaller than the marginal market price, the buyer decreases her bid. If the marginal valuation equals the marginal market price, the buyer updates her bid, which can be in any direction, in an effort to maintain equality between the marginal valuation and marginal market price. Therefore, the dynamics is a tâtonament process

[18], in which each user greedily adapts her bid b_i in the direction of maximizing her payoff, for $\boldsymbol{b}_{-i}$ fixed. The interpretation of the behavior of the buyers is that they can be thought to be strategic on a short time scale, but they are assumed not to exhibit strategic behavior over a long time scale during participation in the decentralized algorithm.

Let $b_i(t)$ be the bid of buyer i and $x_i(t)$ be the allocation to buyer i at time t. Let $B(t) = \sum_i b_i(t)$ be the sum of all the bids at time t. The decentralized algorithm is specified as follows. The initial value $b_i(0)$ is specified to be β_i, where β_i is a given constant with $\beta_i > 0$. For each buyer i, $b_i(t)$ is required to be an absolutely continuous function of t over finite time intervals. It satisfies the following differential inclusion.

$$\begin{aligned} & b_i(0) = \beta_i \\ & \dot{b}_i(t) \in b_i(t) \cdot \overline{sgn}(U_i'(x_i(t)) - g(B(t))) \end{aligned} \tag{22}$$

where

$$x_i(t) = \frac{b_i(t)}{B(t)} C, \quad \forall\ i, \tag{23}$$

and

$$\overline{sgn}(x) = \begin{cases} \{1\}, & \text{if } x > 0 \\ [-1, 1], & \text{if } x = 0 \\ \{-1\}, & \text{if } x < 0 \end{cases}.$$

Equation (22) is understood to hold at all t such that $\dot{b}(t)$ exists. Since the set-valued function $\overline{sgn}(x)$ is locally bounded, has a closed graph, and has convex set values, a solution of the differential inclusion exists and is known as a Krasovskii solution [10].

An absolutely continuous function is differentiable almost everywhere. A time $t > 0$ is a *regular point* if the time derivatives $\dot{b}(t)$ exists. Without specific notification, we only study the regular points in the following discussion.

An example of the decentralized algorithm is shown in Figure 2. In the example, there are four buyers and their valuation functions are $U_1(x) = \log x$, $U_2(x) = -\frac{1}{x}$, $U_3(x) = \log(x + 1)$ and $U_4(x) = (x+1)^{\frac{1}{2}} - 1$. The capacity C is 3. The initial value of the bid vector β is $(0.1, 0.1, 0.1, 0.1)$. We choose the marginal price function $g(u) = \frac{u^2}{C}$.

During an initial time interval, $g(B(t))$ is less than all the buyers' marginal valuations, so that all the buyers increase their bids proportionally, while their allocations, and hence their marginal valuations, remain constant. The marginal price $g(B(t))$ first hits $U_4'(x_4(t))$ and then hits $U_3'(x_3(t))$. Both $U_3'(0)$ and $U_4'(0)$ are finite. The marginal value $U_4'(x_4(t))$ cannot keep up with the change of $g(B(t))$ and the bid of buyer 4 approaches 0. In contrast, $U_3'(x_3(t))$ is large enough and finally converges to the market clearing price for price taking buyers, λ.

Proposition 4.1 *Suppose the following two conditions are true:*

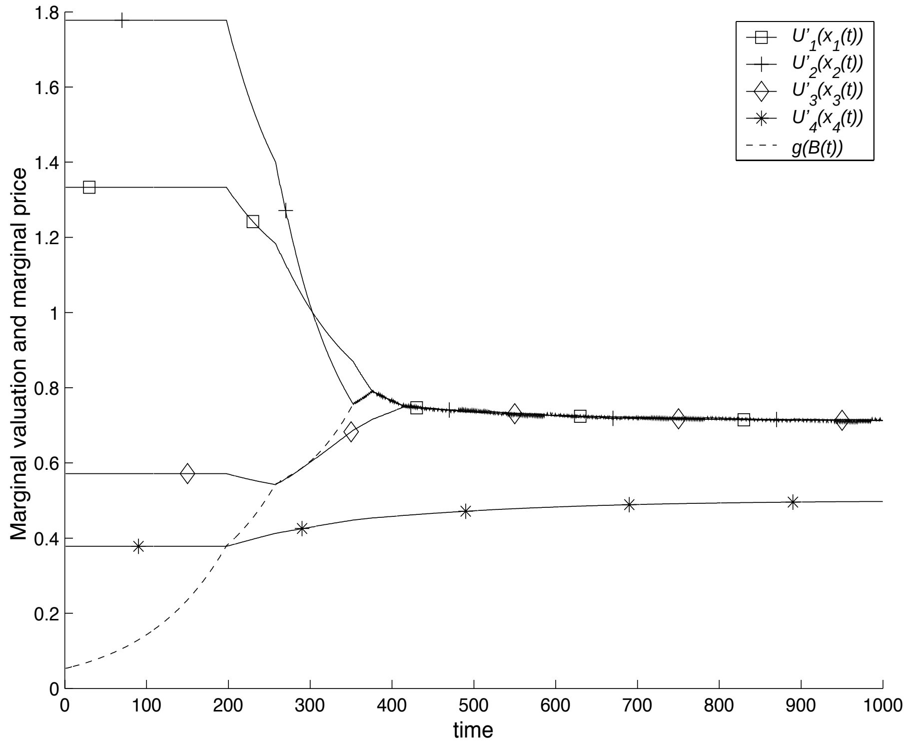


Figure 2: An example of the convergence of a decentralize dynamic system

1. *For all* i*,* $U_i'(x_i)$ *is continuously differentiable on* $(0, C]$ *and* $U_i''(x_i) < -\sigma$ *for some* $\sigma > 0$*.*

2. $g(u)$ *is continuously differentiable.*

There is an efficient NEP $\boldsymbol{b}_\infty$ *such that* $\lim_{t\to\infty} \boldsymbol{b}(t) = \boldsymbol{b}_\infty$ *for any trajectory* $(\boldsymbol{b}(t) : t \geq 0)$ *of the decentralized algorithm.*

Remark 4.1 *Proposition 4.1 does not require Assumption 3.1. If Assumption 3.1 doesn't hold, there are infinitely many NEPs. However, the algorithm in (22) always converges to an efficient NEP as long as all the buyers start with nonzero bids. Therefore, the bluff behavior of the buyers described in Section 3.4 does not emerge.*

Proposition 4.1 is proved in Appendix G. It can be shown that if $V(\boldsymbol{x}) = \sum_{i=i}^{N} U_i(x_i)$, then $V(\boldsymbol{x}_t)$ is nondecreasing in t. Thus, the sum of valuations V has some of the properties of a Lyapunov function. However, V is a function of $\boldsymbol{x}$, whereas the system is specified in terms of a differential inclusion for $\boldsymbol{b}$.

The stability result gives insight for designing decentralized congestion control and resource allocation algorithms in a communication network with strategic buyers. However, there are still restrictions on the algorithm for real application. In the algorithm, all the buyers have to update continuously, synchronously and proportionally. A challenge for future work is to design globally stable algorithms which allow different buyers to update at arbitrary nonzero rates, or to design globally stable algorithms with discrete step-sizes allowing possibly asynchronous updates and delayed feedback signals.

5 Comparison with other implementation mechanisms

The work within the broad literature on implementation theory or mechanism design closest to that in this paper is described in Section 3.7 of the survey paper of Maskin and Sjöström [19], on Nash implementation in the context of economic environments. Much of the literature describes Nash implementations of Walrasian efficient allocations for exchange economies. Hurwicz [11] and Schmeidler [24] constructed simple market mechanisms in which buyers announce prices and allocations, and Postlewaite and Wettstein [21] provided a continuous feasible mechanism. The work of Williams [27] and Reichelstein and Reiter [22] is concerned with the minimum required dimension of the message space needed to implement a social choice function. We will describe two mechanisms suggested to us by this literature and give the results of NEPs of these two mechanisms (Proposition 5.1 and Proposition 5.2). The proof of these two mechanisms are in Appendix H.

We call the first mechanism PQ2, for "price-quantity" with two buyers suggesting prices. Suppose there are at least two buyers. The bid of buyer i for $1 \leq i \leq n-2$ is one-dimensional: $\sigma_i = w_i \in \mathbb{R}_{++}$, and the bids of the other two buyers are two dimensional: $\sigma_{n-1} = (w_{n-1}, a)$ and $\sigma_n = (w_n, b)$, where $\mathbb{R}_{++}$ is the set of strictly positive real numbers. The interpretation is that w_i is an indication of the quantity desired by buyer i, and a and b are suggested prices. The mechanism is given by:

Mechanism PQ2

$$\text{Proportional allocation rule: } x_i(\sigma) = \frac{w_i C}{W}$$
$$\text{Payment rule: } m_i(\sigma) = x_i \overline{p}(\boldsymbol{\sigma})$$

where $W = \sum_i w_i$ and $\overline{p}(\boldsymbol{\sigma}) = \max\{a, b\}$.

Proposition 5.1 *Suppose the valuation functions U_i are strictly concave, and continuously differentiable with $U_i'(0_+) = \infty$. There exist NEPs for mechanism PQ2. For each NEP, the allocation is efficient, and $\overline{p} = \lambda$, where λ is the market clearing price for price taking buyers.*

Note that the allocation and payments for mechanism PQ2 are not changed if the vector $w = (w_1, \ldots, w_n)$ is multiplied by a positive constant, which gives an additional degree of freedom for price setting. This suggests the following rule, which is an adaptation of the rule in Section 2 of [22].

The bid of each buyer i is denoted by σ_i, where $\sigma_i = w_i \in \mathbb{R}_{++}$ for $1 \le i \le n-1$, and $\sigma_n = (w_n, p) \in \mathbb{R}^2_{++}$. As before w_i is an indication of the quantity desired by buyer i, and p is the price suggested by buyer n. The mechanism is given by:

Mechanism PQ1

$$\text{Proportional allocation rule: } x_i(\sigma) = \frac{w_i C}{W}$$
$$\text{Payment rule: } m_i(\sigma) = p_i x_i$$

where

$$p_i = \begin{cases} p, & 1 \le i \le n-1 \\ \left(\sum_{j=1}^{n-1} w_j\right) + \left(p - \sum_{j=1}^{n-1} w_j\right)^2, & i = n \end{cases}.$$

Proposition 5.2 *Suppose the valuation functions U_i are strictly concave, and continuously differentiable with $U_i'(0_+) = \infty$. Then mechanism PQ1 has a unique NEP, the NEP is efficient, and at the NEP, $p_i = \lambda$ for all i.*

Some comments on mechanisms PQ2 and PQ1, comparing them to the g-mechanism studied in this paper, are in order.

1. The dimensions of the aggregate message spaces for mechanisms PQ2, PQ1, and the g-mechanism of this paper, are $n+2$, $n+1$, and n respectively. The prices charged by mechanism PQ2 are uniform, both in and out of equilibrium. The prices charged by mechanism PQ1 are uniform in equilibrium, but not out of equilibrium. The prices of the g mechanism are uniform neither in nor out of equilibrium. The analysis of Reichelstein and Reiter [22], Section 2, shows that $n+1$ is the minimum required dimension for efficient implementation with a uniform price in equilibrium. (The constraint of uniform price in equilibrium is automatic if our problem is embedded into the framework of a Walrasian exchange economy with two commodities: transmission bandwidth and money.) Johari and Tsitsiklis [13] provided quantitative results along the same lines. They showed that the worst case efficiency ratio for uniform price allocation with bids equal payments is 75%. Johari [12] showed that if the price is uniform, then using any mechanism based on one-dimensional bids (within regularity conditions) does not yield a worst case efficiency ratio any higher than the 75% ratio achieved by proportional allocation with bids equal payments.

2. The functions appearing in mechanisms PQ1 and the g-mechanism are continuously differentiable. The price function $\overline{p}$ in mechanism PQ2 is continuous, but only piecewise differentiable. The following more regular price function, requiring three buyers to suggest prices a, b, and c, was suggested by Postlewaite and Wettstein [21]:

$$\overline{p}_{PW}(a,b,c) = \begin{cases} a, & \text{if } a = b = c \\ \frac{a(b-c)^2 + b(a-c)^2 + c(a-b)^2}{(a-b)^2 + (a-c)^2 + (b-c)^2}, & \text{else} \end{cases}.$$

The directional derivatives of the function $\overline{p}_{PW}$ exist in any direction about any point, but at points of the form $a = b = c$ they do not give a linear mapping. (i.e. the function $\overline{p}_{PW}$ is Gâtaeux differentiable but not Fréchet differentiable). We do not know of a continuously Fréchet differentiable allocation mechanism (of any dimension) which charges a uniform price both in and out of equilibrium and implements the efficient allocation. We also do not know whether there exists a Gâtaeux differentiable allocation mechanism of dimension less than $n + 2$ which charges a uniform price both in and out of equilibrium and implements the efficient allocation.

3. The revenue generated by PQ1 and PQ2 is λC, equal to the maximum possible under the conditions of Proposition 3.3. The revenue of the g-mechanism depends on the choice of the g function, and can take any value strictly between zero and λC.

4. Mechanisms PQ1 and PQ2 cause the marginal and average price of each buyer to be the same through an explicit price setting mechanism, whereas the g-mechanism implicitly insures equality of marginal prices. The average price of the g-mechanism depends on the choice of the g function. The explicit price setting of mechanisms PQ1 and PQ2 requires a somewhat different mindset for the one or two buyers suggesting prices. Those buyers need to adjust their suggested prices in order to equate supply and demand, rather than selfishly trying to minimize their own price.

6 Conclusion

Results provided in Section 5 raise an interesting question about the existence of differentiable allocation mechanisms which charge uniform prices, even out of equilibrium. Can they be Fréchet differentiable? Is there a Gâtaeux differentiable one using less than $n + 2$ dimensions?

Our ultimate goal is to design efficient mechanisms for network resource allocation. We show that a one-dimensional bid for each buyer is enough for implementing an efficient Nash equilibrium in the allocation of a single divisible good, or a single link network. It is reasonable to explore how g-mechanisms might work in a network with many links forming a general topology. A simple way to use g-mechanisms in a network is to set up an itemized bid game [9, 13]. In an itemized bid game model, each buyer sends a vector of bids with each component corresponding to a link she occupies. The total rate that a buyer gets is the minimum of the rates she is allocated from the links on her path. We can show that the allocation is efficient at the NEP. Future work is to develop a real world implementation of g-mechanisms in networks, involving software in clients and routers, policies for possible use, graphical user interfaces, etc.

A Proof of general revenue bounding proposition

Proposition 2.1 is proved in this section. In addition, an example is given to illustrate the proposition in case multiple NEPs exist.

For a set of N regular buyers $\mathcal{U}$ with valuation functions $(U_1, \ldots, U_N)$, let $\boldsymbol{x}^*$ be the efficient allocation, and let λ be the uniform market clearing price. Define a new set of N regular buyers $\mathcal{V}$ with valuation functions $(V_1, \ldots, V_N)$ by:

$$V_i(x_i) = \begin{cases} -\frac{1}{2}\epsilon(x_i - x_i^*)^2 + \lambda(x_i - x_i^*) + U_i(x_i^*), & \text{if } 0 \le x_i \le x_i^* \\ U_i(x_i), & \text{if } x_i^* < x_i \le C \end{cases},$$

where $\epsilon > 0$. Clearly, $V_i(x_i)$ is regular and

$$V_i'(x_i) = \begin{cases} \lambda - \epsilon(x_i - x_i^*), & \text{if } 0 \le x_i \le x_i^* \\ U_i'(x_i), & \text{if } x_i^* < x_i \le C \end{cases}$$

Hence, the set of buyers $\mathcal{V}$ also has the efficient allocation $\boldsymbol{x}^*$ and uniform market clearing price λ. For any $\delta > 0$, select $\epsilon \le \frac{2\delta}{C}$ and also ϵ small enough such that $U_i'(x_i) \ge V_i'(x_i)$ for all i and $x_i \le x_i^*$. Then for any i, $U_i(x_i) - U_i(x_i^*) \le V_i(x_i) - V_i(x_i^*)$ for all $x_i \le x_i^*$. Suppose $\boldsymbol{b}^*$ is an efficient NEP derived by the mechanism $(\mathcal{B}, \boldsymbol{x}, \boldsymbol{m})$ and the set of buyers $\mathcal{V}$, which is assumed to exist by Condition 1 in this proposition. Then $\Pi_i^{\mathcal{V}}(b_i; \boldsymbol{b}_{-i}^*) - \Pi_i^{\mathcal{V}}(\boldsymbol{b}^*) \le 0$ for all $b_i \ne b_i^*$. Hence, for all $b_i \ne b_i^*$,

$$\begin{aligned} \Pi_i^{\mathcal{U}}(b_i; \boldsymbol{b}_{-i}^*) - \Pi_i^{\mathcal{U}}(\boldsymbol{b}^*) &= U_i(x_i(b_i; \boldsymbol{b}_{-i})) - m_i(b_i; \boldsymbol{b}_{-i}) - (U_i(x_i(\boldsymbol{b}^*)) - m_i(\boldsymbol{b}^*)) \\ &= U_i(x_i(b_i; \boldsymbol{b}_{-i})) - U_i(x_i(\boldsymbol{b}^*)) - m_i(b_i; \boldsymbol{b}_{-i}) + m_i(\boldsymbol{b}^*) \\ &\le V_i(x_i(b_i; \boldsymbol{b}_{-i})) - V_i(x_i(\boldsymbol{b}^*)) - m_i(b_i; \boldsymbol{b}_{-i}) + m_i(\boldsymbol{b}^*) \\ &= \Pi_i^{\mathcal{V}}(b_i; \boldsymbol{b}_{-i}^*) - \Pi_i^{\mathcal{V}}(\boldsymbol{b}^*) \le 0, \end{aligned}$$

which implies that $\boldsymbol{b}^*$ is an NEP for $\mathcal{U}$ using mechanism $(\mathcal{B}, \boldsymbol{x}, \boldsymbol{m})$ (We've basically used the fact that the original valuation vector $(U_1, \ldots, U_N)$ is a monotonic transform of the valuation vector $(V_1, \ldots, V_N)$ [19]).

Meanwhile, by the Condition 2 there exists a bid $\boldsymbol{b}^o$ so that $x_i(b_i^o; \boldsymbol{b}_{-i}) = 0$ and $m_i(b_i^o; \boldsymbol{b}_{-i}) = 0$ for all $\boldsymbol{b}_{-i}$. Then $\Pi_i^{\mathcal{V}}(\boldsymbol{b}^*) \ge \Pi_i^{\mathcal{V}}(b_i^o; \boldsymbol{b}_{-i}^*)$ because $\boldsymbol{b}^*$ is an NEP. Therefore, $V_i(x_i^*) - m_i(\boldsymbol{b}^*) \ge V_i(0)$, which implies

$$m_i(\boldsymbol{b}^*) \le V_i(x_i^*) - V_i(0) = \lambda x_i^* + \frac{1}{2}\epsilon x_i^{*2}.$$

Therefore, $p_i^* \le \lambda + \frac{1}{2}\epsilon x_i^* \le \lambda + \delta$. The proof of Proposition 2.1 is complete.

Example A.1 *Suppose $N = C = 2$ and consider the mechanism $(\mathcal{B}, \boldsymbol{x}, \boldsymbol{m})$ such that the bid of* *buyer i is $\boldsymbol{\sigma}_i = (a_i, b_i) \in \{0,1\} \times \mathbb{R}_+$, and, for $i = 1, 2$:*

$$x_i(\boldsymbol{b}) = \begin{cases} \frac{b_i C}{b_1+b_2}, & \text{if } a_1 = a_2 = 0, \quad b_1 + b_2 > 0 \\ 0, & \text{if } a_1 = a_2 = 0, \quad b_1 + b_2 = 0 \\ 0, & \text{if } a_1 \neq a_2 \\ 1, & \text{if } a_1 = a_2 = 1 \end{cases}$$

$$m_i(\boldsymbol{b}) = \begin{cases} b_1 b_2, & \text{if } a_1 = 0 \text{ or } a_2 = 0 \\ 2.5, & \text{if } a_1 = a_2 = 1 \end{cases}$$

Intuitively, a_i indicates which game buyer i would like to play. If $a_1 = a_2 = 1$, the buyers agree to evenly split the capacity for payments of 2.5 each. If $a_1 \neq a_2$ the allocations are zero. If the buyers were restricted to bids such that $a_1 = a_2 = 0$, then, as a function of b_1 and b_2, this mechanism reduces to a simple g-mechanism as described in Section 3, and therefore for any two buyers there would exist a unique NEP b_1^*, b_2^* and the allocation would be efficient. Without any restriction on a_1 and a_2, the pair $\boldsymbol{\sigma_1} = (0, b_1^*)$, $\boldsymbol{\sigma_2} = (0, b_2^*)$ is an NEP with the same allocation and revenue, because neither buyer would have incentive to change her a_i value to one. Thus, the mechanism satisfies the assumptions of Proposition 2.1.

Consider two buyers with common valuation functions $U_1(x) = U_2(x) = 4\log(x+1)$ and suppose $C = 2$. The efficient allocation is $\boldsymbol{x}^* = (1, 1)$ and $\lambda = 2$. Thus, $\lambda C = 4$. Also, $b_1^* = b_2^* = 1$, so that pair $(0,1), (0,1)$ is an NEP with the efficient allocation, and revenue 2. But the pair $(1,1), (1,1)$ is also an NEP with the efficient allocation, and its revenue is 5. The example shows that, under the conditions of Proposition 2.1, the revenue can be larger than λC at some efficient NEPs.

B Proof of existence, uniqueness and efficiency of a NEP in *g*-mechanism

Proposition 3.1 is proved in this appendix. The proof has four parts. First, if $\boldsymbol{b} = 0$, then fix a buyer i and choose $\bar{b}_i$ being any positive ϵ. Then $\Pi(\bar{b}_i; \boldsymbol{b}_{-i}) = U_i(C) - m_i(\epsilon; \boldsymbol{0}_{-i}) = U_i(C)$. So $\Pi(\bar{b}_i; \boldsymbol{b}_{-i}) - \Pi(b_i; \boldsymbol{b}_{-i}) = U_i(C) - U_i(0) > 0$ since U_i is strictly increasing. Thus $\boldsymbol{b} = 0$ is not an NEP.

Second, we show that if exactly one coordinate of $\boldsymbol{b}$ is positive, then $\boldsymbol{b}$ is not an NEP. Consider $\boldsymbol{b}$ such that $b_i > 0$ and $b_j = 0$ for all $j \neq i$. From (15) and (14), we derive, for $j \neq i$,

$$\frac{\partial \Pi_j}{\partial b_j}(\bar{b}_j; \boldsymbol{b}_{-j}) = C \frac{b_i}{(\bar{b}_j + b_i)^2} \left[U_j'\left(\frac{\bar{b}_j}{\bar{b}_j + b_i} C \right) - g(\bar{b}_j; \boldsymbol{b}_{-j}) \right]. \tag{24}$$

Thus,

$$\frac{\partial \Pi_j}{\partial b_j}(b_i; \boldsymbol{0}_{-i}) = \frac{C}{b_i} \left[U_j'(0) - g(b_i; \boldsymbol{0}_{-i}) \right], \quad \forall\, j \neq i. \tag{25}$$

By Assumption 3.1 at least two buyers have marginal valuation $+\infty$ at 0, so there must be a buyer k different from buyer i such that $U_k'(0) = +\infty$. Thus, $\frac{\partial \Pi_k}{\partial b_k}(0; \boldsymbol{b}_{-k}) = +\infty$, which implies that $\boldsymbol{b}$ is not an NEP. Hence, any NEP $\boldsymbol{b}$ must have at least two positive coordinates.

Third, consider a $\boldsymbol{b}$ with at least two positive coordinates. For all i, from (15) and (14),

$$\frac{\partial \Pi_i}{\partial b_i}(\bar{b}_i; \boldsymbol{b}_{-i}) = \frac{CB_{-i}}{(\bar{b}_i + B_{-i})^2} \left[U_i'(x_i) - g(\bar{b}_i; \boldsymbol{b}_{-i}) \right]. \tag{26}$$

The quantity in square brackets in (26) is strictly decreasing in $\bar{b}_i$, so that $\Pi_i(\bar{b}_i; \boldsymbol{b}_{-i})$ is a strictly quasi-concave function of $\bar{b}_i$. There exists a unique value b_i of $\bar{b}_i$ maximizing $\Pi_i(\bar{b}_i; \boldsymbol{b}_{-i})$ for fixed $\boldsymbol{b}_{-i}$. Furthermore, for i fixed, Π_i is maximized with respect to b_i for $\boldsymbol{b}_{-i}$ fixed if and only if either $U_i'(x_i) = g(\boldsymbol{b})$ or $U_i'(0) \leq g(\boldsymbol{b})$ and $b_i = 0$. Thus (18) is a necessary and sufficient condition for $\boldsymbol{b}$ to be an NEP.

Fourth, we prove the existence, uniqueness, and efficiency of the NEP. Let $(\boldsymbol{x}^*, \lambda)$ be the unique solution to (2) . Select θ^* so that $g(\theta^* \boldsymbol{x}^*) = \lambda$ and let $\boldsymbol{b}^* = \theta^* \boldsymbol{x}^*$. Then $(\boldsymbol{x}^*, \boldsymbol{b}^*)$ is a solution to (18), so $\boldsymbol{b}^*$ is an NEP and is efficient. Conversely, if $\boldsymbol{b}^*$ is an NEP, then (18) holds. Setting $\lambda = g(\boldsymbol{b}^*)$, it follows from (18) that $(\boldsymbol{x}(\boldsymbol{b}^*), \lambda)$ is a solution to (2). Since the vector $(\boldsymbol{x}^*, \lambda)$ satisfying (2) is unique, $\boldsymbol{b}^*$ is also unique as the equality $g(\theta \boldsymbol{x}^*) = \lambda$ has a unique solution.

The proof of Proposition 3.1 is complete.

C Proof of the generality result

Proposition 3.2 is proved in this appendix.

Lemma C.1 *Let $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and let K be a relatively closed subset of $\mathbb{R}_{++}$ such that the restriction of f to K is a one-to-one onto map of K onto $\mathbb{R}_{++}$. Then $K = \mathbb{R}_{++}$ and f is strictly monotone.*

Lemma C.2 *Suppose the conditions of Lemma C.1 hold. Given $0 < a < b$, let $K_{ab} = K \cap f^{-1}([a, b])$. If $\inf(\mathbb{R}_{++} - K_{a,b}) = 0$ and $\sup(\mathbb{R}_{++} - K_{ab}) = +\infty$ then K_{ab} is a closed interval and f is strictly monotone on that interval.*

Proof of Lemma C.2 Let $(u_n : -\infty < n < \infty)$ be a doubly infinite strictly increasing sequence such that $u_n \in \mathbb{R}_{++} - K_{ab}$ for all n, with $\lim_{n \to -\infty} u_n = 0$ and $\lim_{n \to \infty} u_n = +\infty$. Such sequences exist by the assumptions of the lemma. Note that $K_{ab} \cap (u_n, u_{n+1})$ is a compact set for each n, because it is equal to $K_{ab} \cap [u_n, u_{n+1}]$. Moreover, these sets are disjoint and their union is K_{ab}. The images of these sets under f are disjoint, compact, and their union is the interval $[a, b]$. A simple form of Sierpinski's theorem [3] [6] states that if an interval is equal to the union of countably many disjoint closed sets, then one of the sets must be the interval itself. Applying this theorem yields that $K_{ab} \cap (u_n, u_{n+1})$ is nonempty for at most one value of n. By the flexibility in the choice of (u_n), it follows that K_{ab} is a closed interval. Since f restricted to that interval is continuous and one-to-one, f must be strictly monotone on the interval. □

[6] Acknowledgment: The authors are grateful to Prof. Slawomir Solecki for pointing them to Sierpinski's theorem.

Proof of Lemma C.1 It is enough to show that the conditions of Lemma C.2 hold for all choices of a, b with $0 < a < b$. That is, we will show that if $0 < a < b$ and if $u = \inf(\mathbb{R}_{++} - K_{a,b})$ and $v = \sup(\mathbb{R}_{++} - K_{ab})$, then $u = 0$ and $v = +\infty$. For the sake of argument by contradiction, suppose that $u > 0$ (the case $v < \infty$ is similar). Then $(0, u) \subset K_{ab}$. But $f^{-1}([a, b])$ is a relatively closed subset of $\mathbb{R}_{++}$, so that K_{ab} is a relatively closed subset of $\mathbb{R}_{++}$, and therefore $(0, u] \subset K_{ab}$. Thus, f restricted to the interval $(0, u]$ is strictly monotone and has values in $[a, b]$. It follows that $\lim_{x\to 0} f(x) = y^*$ for some $y^* \in [a, b]$. In particular, $y^* \neq 0$. Since f maps $\mathbb{R}_{++}$ onto $\mathbb{R}_{++}$, it follows that $\liminf_{x\to\infty} f(x) = 0$ and $\limsup_{x\to\infty} f(x) = +\infty$. Therefore, the conditions of Lemma C.2 apply for any $\overline{a}$ and $\overline{b}$ with $0 < \overline{a} < \overline{b} < y^*$. It follows that f is ultimately monotone decreasing with limit 0 as $x \to \infty$, but then it is impossible for the range of f to be all of $\mathbb{R}_{++}$, which is contradiction. The lemma is proved. □

Proof of Proposition 3.2 Let $\mathcal{D} = \{\boldsymbol{b} \in \mathbb{R}_{++} : g_1(\boldsymbol{b}) = \cdots = g_N(\boldsymbol{b})\}$. We claim that $\mathcal{D} = \mathbb{R}^N_{++}$. Fix $\boldsymbol{x} \in \mathbb{R}^N_{++}$ with $\sum_i x_i = C$, and let $\mathcal{D}_{\boldsymbol{x}} = \{\theta > 0 : \theta\boldsymbol{x} \in \mathcal{D}\}$. To prove the claim is suffices to show that $\mathcal{D}_{\boldsymbol{x}} = \mathbb{R}_{++}$, because $\mathcal{D}$ is the union of the sets of the form $\mathcal{D}_{\boldsymbol{x}}$. Let f be defined on $\mathbb{R}_{++}$ by $f(\theta) = g_1(\theta x)$ for $\theta > 0$. Then f is continuous. Given any $\lambda > 0$, there exists a choice of valuation functions $U_1, \ldots, U_N$ so that $\boldsymbol{x}$ is the efficient allocation, and λ is the market clearing price for price taking buyers. By Assumption G3, there is a unique NEP for this set of buyers, and the NEP is efficient. Thus, the NEP $\boldsymbol{b}$ must have the form $\boldsymbol{b} = \theta\boldsymbol{x}$, for some unique value of $\theta > 0$. That is, there exists precisely one value of $\theta \in \mathcal{D}_{\boldsymbol{x}}$ such that $f(\theta) = \lambda$. Therefore, f and the relatively closed subset $\mathcal{D}_{\boldsymbol{x}}$ of $\mathbb{R}_{++}$ satisfy the conditions of Lemma C.1. It follows that $\mathcal{D}_{\boldsymbol{x}} = \mathbb{R}_{++}$, so $\mathcal{D} = \mathbb{R}^N_{++}$ as claimed.

Since the functions $g_1, \ldots, g_N$ are equal over all of $\mathbb{R}^N_{++}$, we can define g to agree with these functions over $\mathbb{R}^N_{++}$. By continuity of the functions g_i over their domains, we can define g over all of $\mathbb{R}_+ - \{\mathbf{0}\}$ so that g restricted to the domain of g_i agrees with g_i for each i. Finally, we set $g(\mathbf{0}) = 0$. The required properties of g are easily checked. The expression for m_i comes from integrating each side of (19). □

D Proofs of properties of g-mechanisms

The results of Section 3.3 are proved in this appendix. The following lemma provides a useful representation of the prices.

Lemma D.1 *If $b_i > 0$ and $\boldsymbol{b}_{-i} \neq \mathbf{0}$, then the price can be represented as*

$$p_i(\boldsymbol{b}) = \int_0^1 g(\beta_i(u), \boldsymbol{b})du, \tag{27}$$

where $\beta_i(u,\boldsymbol{b}) = \frac{B_{-i}b_i u}{B_{-i}+b_i(1-u)}$. *If the marginal price is* $g(B)$ *(i.e. a function of* B *alone) then*

$$p_i(\boldsymbol{b}) = \int_0^1 g(s(u,b_i,B_{-i}))du, \tag{28}$$

where $s(u,b_i,B_{-i}) = \beta_i(u,\boldsymbol{b}) + B_{-i} = \frac{BB_{-i}}{B-ub_i}$.

Proof. In view of the definitions of p_i, m_i, and x_i, (27) is equivalent to

$$\frac{B_iB}{b_i}\int_0^{b_i}\frac{g(t,\boldsymbol{b}_{-i})}{(t+B_{-i})^2}dt = \int_0^1 g(\beta_i(u,\boldsymbol{b}),\boldsymbol{b}_{-i})du. \tag{29}$$

The variable t in the left hand side of (29) and the value of $\beta_i(u,\boldsymbol{b})$ in the right hand side of (29) both range over the interval $[0,b_i]$. Thus, equality holds in (29) if it holds for functions g of the form $g(t,\boldsymbol{b}_{-i}) = I_{\{t\le v\}}$, for any $v\in[0,b_i]$). For such a function g, the left hand side of (29) is given by

$$\frac{B_iB}{b_i}\int_0^{v}\frac{1}{(t+B_{-i})^2}dt = \frac{Bv}{b_i(B_{-i}+v)}$$

and the right hand side of (29) is $\beta_i^{-1}(v,\boldsymbol{b})$, where β_i^{-1} is the inverse of the function $\beta_i(u,\boldsymbol{b})$ for $\boldsymbol{b}$ fixed. It is easy to derive that $\beta_i^{-1}(v,\boldsymbol{b}) = \frac{Bv}{b_i(B_{-i}+v)}$, so the equality holds as required. This proves (27), which in turn implies (28). □

Proof of Proposition 3.3 The lower bound of the price is trivial. By the representation (28) of Lemma D.1, $p_i(\boldsymbol{b})$ is a weighted average of g over the interval $[B_{-i},B]$, and hence $p_i(\boldsymbol{b}) < g(B) = \lambda$.

Next, consider $g_\alpha(u) = \frac{u^\alpha}{C}$ for $\alpha > 0$. The corresponding NEP $\boldsymbol{b}$ satisfies $B^\alpha = \lambda C$, $b_i = r_iB$, and $B_{-i} = (1-r_i)B$, where r_i is determined by $x_i = Cr_i$. Note that λ, C, x_i, and r_i depend on the valuation functions and the capacity, but not on α. The equilibrium price, $p_i^* = m_i/x_i$, can be easily written in terms of r_i, λ and α to yield:

$$p_i^* = \frac{\lambda(1-r_i)}{r_i}\cdot\frac{1-(1-r_i)^{\alpha-1}}{\alpha-1} \tag{30}$$

It is easy to verify that $\lim_{\alpha\to 0} p_i^* = \lambda$ and $\lim_{\alpha\to\infty} p_i^* = 0$ for all i. □

Proof of Proposition 3.4 Proposition 3.4 follows immediately from Lemma D.1 and the fact that for each $u\in[0,1]$ fixed, $\beta_i(u,\boldsymbol{b})$, and hence $g(\beta_i(u,\boldsymbol{b}),\boldsymbol{b}_{-i})$, are increasing functions of $\boldsymbol{b}$. □

Proof of Proposition 3.5 Note that $s(u,b_i,B_{-i}) = \frac{B(B-b_i)}{B-ub_i}$, so that for u and B fixed, s_i is a decreasing function of b_i. Therefore, if $\boldsymbol{b}$ is a bid vector and $0 < b_i < b_j$ for some buyers i and j, then $s(u,b_i,B_{-i}) > s(u,b_j,B_{-j}))$ for $0\le u\le 1$. Therefore, in view of (28), $p_i(\boldsymbol{b}) > p_j(\boldsymbol{b})$. Example 3.1 shows that the volume discounts property does not hold in general for g-mechanisms. □

Proof of Proposition 3.6 Let B, B_{-i}, b_i, and x_i correspond to the NEP for n buyers, and

$\widetilde{B}$, $\widetilde{B}_{-i}$, $\widetilde{b}_i$, and $\widetilde{x}_i$ correspond to the NEP with the $n+1^{st}$ buyer also included. By the efficiency condition (2), the market clearing price λ increases as a new buyer joins in. Therefore $\widetilde{B} > B$ and $\widetilde{x}_i < x_i$ for $1 \le i \le n$. Since $B_{-i} = \frac{B(C-x_i)}{C}$, it follows that $\widetilde{B}_{-i} > B_{-i}$. (However, $\widetilde{b}_i$ can be larger than, smaller than, or equal to, b_i.)

In view of in view of (28), Proposition 3.6 will follow if it can be proved that $s(u, b_i, B_{-i}) \le s(u, \widetilde{b}_i, \widetilde{B}_{-i})$. Equivalently, it is enough to show that

$$BB_{-i}\{\widetilde{B} - u\widetilde{b}_i\} \le \widetilde{B}\widetilde{B}_{-i}\{B - ub_i\}. \tag{31}$$

Since each side of (31) is affine in u, it suffices to verify (31) for $u = 0$ and $u = 1$. The equation is true for $u = 0$ since $B_{-i} \le \widetilde{B}_{-i}$ and the equation is true for $u = 1$ since $B \le \widetilde{B}$. Example 3.2 shows that the price increase property does not hold in general for g-mechanisms. Proposition 3.6 follows. □

E Proofs of results if Assumption 3.1 does not hold

The results of Section 3.4 are proved in this appendix.

Proof of Proposition 3.7 We don't prove the proposition sequentially as it is stated. If Assumption 3.1 holds, we have already proved the uniqueness of b^* and shown that b^* is the unique and efficient NEP in Proposition 3.1. We will prove the proposition in the case that Assumption 3.1 fails to hold.

Suppose in the remainder of the proof that Assumption 3.1 doesn't hold. Also in the remainder of the proof, let $\boldsymbol{b}^*$ denote the unique solution to (18). Such $\boldsymbol{b}^*$ exists, it is an NEP, and the allocation at $\boldsymbol{b}^*$ is efficient. (These properties of (18) and $\boldsymbol{b}^*$ were proved in part four of the proof of Proposition 3.1, which did not require Assumption 3.1). Next, we seek the necessary and sufficient condition for an NEP. As in the proof of Proposition 3.1, we fix a bid vector $\boldsymbol{b}$ and treat three cases: all the components in $\boldsymbol{b}$ are zero, exactly one component in $\boldsymbol{b}$ is positive, and at least two components in $\boldsymbol{b}$ are positive.

Firstly, $\boldsymbol{b} = 0$ is not an NEP by the same argument used in the proof of Proposition 3.1.

Secondly, consider $\boldsymbol{b}$ such that $b_i > 0$ and $b_j = 0$ for all $j \ne i$. One one hand, if $j \ne i$, the partial derivative of her payoff with respect to b_j at $\boldsymbol{b}$ is given by (25). For all $j \ne i$, the payoff of buyer j is maximized at $b_j = 0$ if and only if $g(b_i; \mathbf{0}_{-i}) \ge U'_j(0)$. On the other hand, buyer i has no incentive to change her bid since her payoff is fixed at $U_i(C)$ as long as $b_i > 0$, and is $U_i(0)$ if $b_i = 0$. Hence, $\boldsymbol{b}$ is an NEP if and only if $\boldsymbol{b} \in \hat{\mathcal{B}}_i$. The set of NEPs with one positive component is $\hat{\mathcal{B}} = \cup_i \hat{\mathcal{B}}_i$.

Thirdly, if at least two coordinates of $\boldsymbol{b}$ are positive, then the payoff function of buyer i has the form in (15) and its partial derivative is given by (26). Thus, $\boldsymbol{b}$ is an NEP if and only if it is the solution to (18), i.e. if and only if $\boldsymbol{b} = \boldsymbol{b}^*$.

If there exists a dominant buyer $\bar{i}$, then the efficient allocation is $x^*_{\bar{i}} = C$ and $x^*_j = 0$ for all $j \neq \bar{i}$. From (18), $\boldsymbol{b}^*$ satisfies $g(b^*_{\bar{i}}; \mathbf{0}_{-i}) \geq U'_j(0)$ and $b^*_j = 0$ for all $j \neq \bar{i}$. Hence, $\boldsymbol{b}^* \in \hat{\mathcal{B}}_{\bar{i}}$. Furthermore, all elements in $\hat{\mathcal{B}}_i$ are efficient NEPs, and no other NEPs exist.

If no dominant buyer exists, then at least two components in $\boldsymbol{b}^*$ are positive. Thus $\boldsymbol{b}^* \notin \hat{\mathcal{B}}$ and $\hat{\mathcal{B}} \cup \{\boldsymbol{b}^*\}$ is the set of all NEPs. None of the bid vectors in $\hat{\mathcal{B}}$ yield efficient allocations because efficient allocation has at least two nonzero elements. □

Proof of Proposition 3.8 Note that $U'_I(0) = U'_{I\!I}(0) = +\infty$. From Proposition 1, at the NEP of the ϵ-extended game, $b_I > 0,\ b_{I\!I} > 0$ and

$$\begin{cases} U'_i(x_i(\epsilon)) = \frac{\epsilon w}{x_I(\epsilon)} = \frac{\epsilon(1-w)}{x_{I\!I}(\epsilon)} = \lambda(\epsilon), & \text{if } x_i > 0, \\ U'_i(0) \leq \lambda(\epsilon), & \text{if } x_i = 0. \end{cases} \tag{32}$$

We claim that $\lambda(\epsilon)$ is an increasing function of ϵ. For the sake of proof by contradiction, suppose $0 < \epsilon_1 < \epsilon_2$ and $\lambda(\epsilon_1) > \lambda(\epsilon_2)$. By (32), $x_i(\epsilon_1) \leq x_i(\epsilon_2)$ for all $1 \leq i \leq N$ and $x_I(\epsilon_1) < x_I(\epsilon_2)$ and $x_{I\!I}(\epsilon_1) < x_{I\!I}(\epsilon_2)$. This contradicts the fact that $\sum_{i=1}^N x_i(\epsilon) + x_I(\epsilon) + x_{I\!I}(\epsilon) = C$ for all $\epsilon > 0$. Therefore $\lambda(\epsilon)$ is an increasing function as claimed. Also note that $\lambda(\epsilon) > \min_i\{U'_i(C)\}$. Thus $\lim_{\epsilon\to 0} \lambda(\epsilon) = \lambda > 0$, which implies that $\lim_{\epsilon\to 0} x_I(\epsilon) = 0$ and $\lim_{\epsilon\to 0} x_{I\!I}(\epsilon) = 0$. Since $U'_i(x_i)$ is a strictly decreasing continuous function of x_i for all i, the solution pair $(\boldsymbol{x}(\epsilon), \lambda(\epsilon))$ of (32) converges to the solution pair $(\boldsymbol{x}, \lambda)$ of (2). □

F Absolute continuity

In this section, we review some facts about absolutely continuous functions, which will be used to derive the absolute continuity of $x_i(t)$ and $U'_i(x_i(t))$ for the algorithm of Section 4. A function $f : [a, b] \to \mathbb{R}$ is defined to be absolutely continuous if for any $\epsilon > 0$, there exists a $\delta > 0$, so that for any n, if $a \leq a_1 \leq b_1 \leq \cdots \leq a_n \leq b_n \leq b$ with $\sum_{i=i}^n b_i - a_i \leq \delta$, then $\sum_{i=1}^n |f(a_i) - f(b_i)| \leq \epsilon$. An important fact is that f is absolutely continuous on $[a, b]$ if and only if its derivative f' exists almost everywhere on the interval, and $f(t) = \int_a^t f'(s)ds$ for $t \in [a, b]$ (see [23]). The following facts will also be used:

Fact F.1 *If both f and g are absolutely continuous functions on $[a, b]$, then $f + g$ and $f \cdot g$ are absolutely continuous on $[a, b]$.*

Fact F.2 *If f is absolutely continuous and $|f| > 0$ on $[a, b]$, then $\frac{1}{f}$ is also absolutely continuous on $[a, b]$.*

Fact F.3 *If f is absolutely continuous on $[c, d]$ and g is absolutely continuous with $c \leq g \leq d$ on $[a, b]$, then $f \circ g$ is absolutely continuous on $[a, b]$.*

The above three facts are exercises in [23].

Fact F.4 *If f_l is absolutely continuous on $[a,b]$ for all $1 \le l \le L$, then both $\max_l f_l$ and $\min_l f_l$ are absolutely continuous on $[a,b]$. Therefore, there is a subset I of $[a,b]$ with measure zero, such that f_l for $1 \le l \le L$, $\max_l f_l$, and $\min_l f_l$ are differentiable at all points of $[a,b] \setminus I$. Furthermore, for $x \in [a,b] \setminus I$*

$$\frac{d}{dx} \max_l f_l(x) = f'_m(x) \quad \textit{for } m \in \operatorname*{argmax}_l f_l(x) \tag{33}$$

and

$$\frac{d}{dx} \min_l f_l(x) = f'_n(x) \quad \textit{for } n \in \operatorname*{argmin}_l f_l(x) \tag{34}$$

Proof. The proof is based on the definition of absolute continuity. Since f_l is absolutely continuous for all l, for every $\epsilon > 0$, there exists δ such that for any set of disjoint intervals $[x_i, y_i]$ satisfying $\sum_{i=1}^{N} |x_i - y_i| < \delta$, the following holds for all l: $\sum_{i=1}^{N} |f_l(x_i) - f_l(y_i)| < \frac{\epsilon}{L}$.

Hence,

$$\begin{aligned}
\sum_{i=1}^{N} |\max_l f_l(x_i) - \max_l f_l(y_i)| &\le \sum_{i=1}^{N} \max_l |f_l(x_i) - f_l(y_i)| \\
&\le \sum_{i=1}^{N} \sum_{l=1}^{L} |f_l(x_i) - f_l(y_i)| < L \cdot \frac{\epsilon}{L} = \epsilon
\end{aligned}$$

This proves the absolute continuity of $\max_l f_l$. Similarly, $\min_l f_l$ is absolutely continuous. Hence, the subset I exists. It is assumed that $a, b \in I$.

Let $x \in [a,b] \setminus I$. Then the derivative of $\max_l f_l$ at x must equal its left hand derivative (which is $\min\{f'_m(x) : f_m(x) = \max_l f_l(x)\}$) and must equal its right hand derivative (which is $\max\{f'_m(x) : f_m(l) = \max_l f_l(x)\}$). This implies (33). Equation (34) is proved similarly. □

G Proof of convergence of the algorithm

Proposition 4.1 is proved in this appendix. By the absolute continuity of $b_i(t)$ for all i and the facts about absolutely continuity above, some basic characteristics of the dynamic system can be observed.

Lemma G.1 *$\max_i U'_i(x_i(t))$ and $\min_i U'_i(x_i(t))$ are absolutely continuous over $[0,T]$ for each $T > 0$. Furthermore,*

$$\frac{d}{dt} \max_i U'_i(x_i(t)) = U''_m(x_m(t))\dot{x}_m(t) \quad \textit{for } m \in \operatorname*{argmax}_i U'_i(x_i(t)) \quad \textit{a.e} \tag{35}$$

and

$$\frac{d}{dt} \min_i U'_i(x_i(t)) = U''_n(x_n(t))\dot{x}_n(t) \quad \textit{for } n \in \operatorname*{argmin}_i U'_i(x_i(t)) \quad \textit{a.e} \tag{36}$$

Proof. By the Gronwall-Bellman inequality [15], the fact $b_i(0) > 0$ and $\dot{b}_i(t) \geq -b_i(t)$ imply that $b_i(t) \geq b_i(0)e^{-t} > 0$ for all $t \geq 0$. Hence, $B(t) = \sum_i b_i(t) > 0$ over $[0, T]$. From Facts F.1 and F.2, $x_i(t) = \frac{b_i(t)}{B(t)}C$ is absolutely continuous over $[0, T]$. From Facts F.3 and F.4, $\max_i U_i'(x_i(t))$ and $\min_i U_i'(x_i(t))$ are absolutely continuous over $[0, T]$, and (35) and (36) hold. □

A time $t > 0$ is a *regular point* if the time derivatives at time t of both $\max_i U_i'(x_i)$ and $\min_i U_i'(x_i)$ exist. Without specific notification, we only study the regular points in the following discussion.

Define $\Sigma = \{\boldsymbol{b} : g(B) \in [\min_i U_i'(x_i), \max_i U_i'(x_i)]\}$.

Lemma G.2 *The set $\{t : \boldsymbol{b}(t) \in \Sigma\}$ is nonempty, and $\boldsymbol{b}(t) \in \Sigma$ for all $t \geq t_0$, where $t_0 = \inf\{t : \boldsymbol{b}(t) \in \Sigma\}$.*

Proof. If $\boldsymbol{b}(0) \in \Sigma$, $t_0 = 0$. Else if $g(B(0)) > \max_i U_i'(x_i(0))$, then $\dot{\boldsymbol{b}}_i(t) = -\boldsymbol{b}_i(t), 0 \leq t < t_0$. Furthermore, over $0 \leq t < t_0$, we have $B(t) = B(0)e^{-t}$, $x_i(t) = x_i(0) > 0$, and $U_i'(x_i(t))$ equals $U_i'(x_i(0)) > 0$. Therefore, if $t_0 = +\infty$, then $g(B(t))$ would approach 0 asymptotically. Thus, $g(B(t))$ would meet one of the $U_i'(x_i(t))$ in finite time contradicting the assumption $t_0 = +\infty$. Thus, $t_0 < +\infty$ if $g(B(0)) > \max_i U_i'(x_i(0))$. If $g(B(0)) < \min_i U_i'(x_i(0))$, the same conclusion can be derived since up to time t_0, $U_i'(x_i(t))$ is a constant while $g(B(t))$ approaches $+\infty$. This proves the first part of the lemma.

Suppose there is a $t_2 > t_0$ such that $\boldsymbol{b}(t_2) \notin \Sigma$. Without loss of generality, we can suppose $g(B(t_2)) > \max_i U_i'(x_i(t_2))$. Define

$$t_1 = \sup\{t : \boldsymbol{b}(t) \in \Sigma \text{ and } t_0 \leq t < t_2\}$$

Let $j \in \operatorname{argmax}_i U_i'(x_i(t_1))$. Then $g(B(t_2)) - U_j'(x_j(t_2)) > 0$ and $g(B(t_1)) - U_j'(x_j(t_1)) = 0$. There exists a regular point $\bar{t} \in (t_1, t_2)$ such that $\frac{d}{dt}\left(g(B(\bar{t})) - U_j'(x_j(\bar{t}))\right) > 0$ by the mean value theorem. On the other hand, $\dot{\boldsymbol{b}}(\bar{t}) = -\boldsymbol{b}(\bar{t})$ and $\dot{\boldsymbol{x}}(\bar{t}) = 0$ since $g(B(\bar{t})) > U_i'(x_i(\bar{t})$ for all i. Thus $\frac{d}{dt}\left(g(B(\bar{t})) - U_j'(x_j(\bar{t}))\right) < 0$, which contradicts the conclusion above. □

By Lemma G.2, we can and do assume that the initial value β of the algorithm satisfies $\beta \in \Sigma$ in the following discussions. Therefore, it hold that $\boldsymbol{b}(t) \in \Sigma$ for all $t \geq 0$.

Define

$$\mathcal{G}_+(t) = \{i : U_i'(x_i(t)) > g(B(t))\}$$
$$\mathcal{G}_-(t) = \{i : U_i'(x_i(t)) < g(B(t))\}$$
$$\mathcal{G}_0(t) = \{i : U_i'(x_i(t)) = g(B(t))\}$$

Rewrite the decentralized algorithm as $\dot{b}_i(t) = \xi_i(t)b_i(t)$, where

$$\begin{cases} \xi_i(t) = 1, & \text{if } i \in \mathcal{G}_+(t) \\ \xi_i(t) \in [-1, 1], & \text{if } i \in \mathcal{G}_0(t) \\ \xi_i(t) = -1, & \text{if } i \in \mathcal{G}_-(t) \end{cases} \tag{37}$$

Since $x_i(t) = \frac{b_i(t)}{B(t)}C$,

$$\begin{aligned} \dot{x}_i &= \sum_{j \neq i} -\frac{b_i}{B^2} C \dot{b}_j + \frac{\sum_{j \neq i} b_j}{B^2} C \dot{b}_i \\ &= \frac{C}{B^2} \left(\sum_{j \neq i} (\xi_i - \xi_j) b_i b_j \right) \end{aligned} \tag{38}$$

Lemma G.3 $\max_i U_i'(x_i(t))$ *monotonically decreases and* $\min_i U_i'(x_i(t))$ *monotonically increases.*

Proof. Fix a regular point t. The proof is organized into two cases: $\max_i U_i'(x_i(t)) = \min_i U_i'(x_i(t))$ and $\max_i U_i'(x_i(t)) > \min_i U_i'(x_i(t))$.

If $\max_i U_i'(x_i(t)) = \min_i U_i'(x_i(t))$, then $\frac{d}{dt} \max_i U_i'(x_i(t)) = \frac{d}{dt} \min_i U_i'(x_i(t))$ by Lemma G.1. So by Fact F.4, $\frac{d}{dt} U_i'(x_i(t))$ is the same for all i. These derivatives must all be zero since $\sum_i x_i \equiv C$. Thus, $\frac{d}{dt} \max_i U_i'(x_i(t)) = \frac{d}{dt} \min_i U_i'(x_i(t)) = 0$.

Next we study the case that $\max_i U_i'(x_i(t)) > \min_i U_i'(x_i(t))$. If $i \in \mathcal{G}_+(t)$, $\xi_i = 1$. By (38)

$$\dot{x}_i = \frac{C b_i}{B^2} \left(\sum_{j \in \mathcal{G}_0} (1 - \xi_j) b_j + 2 \sum_{j \in \mathcal{G}_-} b_j \right) \geq 0$$

Similarly, if $i \in \mathcal{G}_-(t)$, then $\dot{x}_i \leq 0$. Consider any $\bar{i} \in \operatorname{argmax}_i U_i'(x_i(t))$. There are two cases, $\bar{i} \in \mathcal{G}_+$ or $\bar{i} \in \mathcal{G}_0$. If $\bar{i} \in \mathcal{G}_+$, we have shown $\dot{x}_{\bar{i}} > 0$. If $\bar{i} \in \mathcal{G}_0(t)$, $\mathcal{G}_+ = \phi \neq \mathcal{G}_-$. Thus, $\sum_{i \in \mathcal{G}_0} \dot{x}_i = -\sum_{i \in \mathcal{G}_-} \dot{x}_i \geq 0$. From Lemma G.1, $\frac{d}{dt} U_i'(x_i(t))$ are equal for all $i \in \mathcal{G}_0$, which implies $\dot{x}_{\bar{i}} \geq 0$. Hence, in both cases, $\frac{d}{dt} \max_i U_i'(x_i(t)) = U_{\bar{i}}''(x_{\bar{i}}(t)) \dot{x}_{\bar{i}}(t) \leq 0$. This implies that $\max_i U_i'(x_i(t))$ is monotonically decreasing. With the same argument, the monotone increase of $\min_i U_i'(x_i(t))$ can be proved. □

Remark G.1 *Since* $\max_i U_i'(x_i(t))$ *and* $\min_i U_i'(x_i(t))$ *are monotone and do not cross, they both converge. Let* $\overline{u} = \lim_{t \to \infty} \max_i U_i'(x_i(t))$ *and* $\underline{u} = \lim_{t \to \infty} \min_i U_i'(x_i(t))$.

Remark G.2 *Since* $g(B(t)) \in [\min_i U_i'(x_i(0)), \max_i U_i'(x_i(0))]$ *for all* $t \in [0, +\infty)$, $B(t)$ *is bounded over* $[0 + \infty)$.

Remark G.3 *If* $j \in \operatorname{argmax}_i U_i'(x_i(t))$, *then* $x_j(\tilde{t}) \geq x_j(t)$ *for all* $\tilde{t} > t$ *by Lemma G.3.*

Lemma G.4 *If* $\dot{B}(t) \leq 0$, *then* $\dot{x}_i(t) \geq x_i(t)$ *for all* $i \in \mathcal{G}_+$.

Proof. Note that $\dot{b}_i(t) = b_i(t)$ for $i \in \mathcal{G}_+$. If $\dot{B}(t) \leq 0$ and $i \in \mathcal{G}_+$, then

$$\dot{x}_i(t) = \frac{\dot{b}_i(t)B(t) - b_i(t)\dot{B}(t)}{B^2(t)} C = x_i(t) \left(1 - \frac{\dot{B}(t)}{B(t)} \right) \geq x_i(t).$$

□

To complete the proof of the convergence of the decentralized system, we first prove the convergence of the marginal price $g(B(t))$. Roughly speaking, once the price g(B(t)) nearly converges, the buyers will be attracted by the almost fixed price and hence their bids must also converge.

Lemma G.5 *The marginal price $g(B(t))$ converges.*

Proof. If $\overline{u} = \underline{u}$, then $\lim_{t\to+\infty} g(B(t)) = \overline{u} = \underline{u}$ from Lemma G.2.

Else if $\overline{u} \neq \underline{u}$, we prove the convergence of $g(B(t))$ by contradiction. Suppose $g(B(t))$ doesn't converge. Let $\overline{g} = \limsup_{t\to+\infty} g(B(t))$ and $\underline{g} = \liminf_{t\to+\infty} g(B(t))$. Clearly $\overline{g}$ and $\underline{g}$ satisfy $\underline{u} \leq \underline{g} < \overline{g} \leq \overline{u}$. Let $T_j = \inf\{t : U_j'(x_j(t)) = \max_i U_i'(x_i(t))\}$ and let $\tilde{\mathcal{N}} = \{j : T_j < +\infty\}$. Define $T_o = \max_{i\in\tilde{\mathcal{N}}} T_i$. Then by Remark G.3,

$$x_j(t) \geq \gamma,\ \forall\ t \geq T_o,\ j \in \tilde{\mathcal{N}} \tag{39}$$

where $\gamma = \min_{j\in\tilde{N}} x_j(T_j)$.

With the assumption that $U_i''(x_i) \leq -\sigma$ for all i, by Lemmas G.1 and G.4, the following holds at every $t > T_o$ satisfying $\dot{B}(t) \leq 0$ and $g(B(t)) < \max_i U_i'(x_i(t))$.

$$\begin{aligned} \frac{d}{dt}\max_i U_i'(x_i(t)) &= U_j''(x_j(t))\dot{x}_j(t),\ \text{where} j \in \operatorname*{argmax}_i U_i'(x_i(t)) \\ &\leq -\sigma\gamma. \end{aligned} \tag{40}$$

For any $\epsilon > 0$, there exists T_ϵ such that if $\bar{t}_1$ and $\bar{t}_2$ are times such that $\bar{t}_2 > \bar{t}_1 > T_\epsilon$, then

$$|\max_i U_i'(x_i(\bar{t}_2)) - \max_i U_i'(x_i(\bar{t}_1))| < \epsilon. \tag{41}$$

By the definition of $\overline{g}$ and $\underline{g}$, there exist $t_2 > t_1 > (T_o \vee T_\epsilon)$ such that $g(B(t_1)) \in (\overline{g} - \delta, \overline{g}]$ and $g(B(t_2)) \in [\underline{g}, \underline{g} + \delta)$, where δ is some positive number less than $\frac{\overline{g}-\underline{g}}{2}$. Hence, there exists an interval $(\tilde{t}_1, \tilde{t}_2) \subset (t_1, t_2)$ such that $g(B(\tilde{t}_1)) = \overline{g} - \delta$, $g(B(\tilde{t}_2)) = \underline{g} + \delta$ and $g(B(t)) \in (\underline{g} + \delta, \overline{g} - \delta)$ for all $\tilde{t}_1 < t < \tilde{t}_2$.

Since $B(t)$ is bounded and $|\frac{d}{dt}g(B(t))| = g'(B(t))|\dot{B}(t)| \leq g'(B(t))B(t)$, $\frac{d}{dt}g(B(t))$ is bounded. We can assume $|\frac{d}{dt}g(B(t))| < \Gamma$ for some $\Gamma > 0$. Therefore,

$$\begin{aligned} g(\tilde{t}_2) - g(\tilde{t}_1) &= \int_{\tilde{t}_1}^{\tilde{t}_2} \frac{d}{dt}g(B(t))dt \\ &\geq \int_{\tilde{t}_1}^{\tilde{t}_2} \frac{d}{dt}g(B(t))I_{\{\dot{B}(t)\leq 0\}}dt \\ &\geq \int_{\tilde{t}_1}^{\tilde{t}_2} (-\Gamma)I_{\{\dot{B}(t)\leq 0\}}dt \end{aligned}$$

which implies $m(\{t : \dot{B}(t) \leq 0, \tilde{t}_1 \leq t \leq \tilde{t}_2\}) \geq \frac{\overline{g}-\underline{g}-2\delta}{\Gamma}$.

Hence,

$$\begin{aligned} |\max_i U_i'(x_i(\tilde{t}_2)) - \max_i U_i'(x_i(\tilde{t}_1))| &= |\int_{\tilde{t}_1}^{\tilde{t}_2} \frac{d}{dt} \max_i U_i'(x_i(t))dt| \\ &\geq |\int_{\tilde{t}_1}^{\tilde{t}_2} \frac{d}{dt} \max_i U_i'(x_i(t)) \cdot I_{\{\dot{B}(t)\leq 0\}} dt| \\ &\geq |\int_{\tilde{t}_1}^{\tilde{t}_2} (-\sigma\gamma) \cdot I_{\{\dot{B}(t)\leq 0\}} dt| \\ &= \frac{\sigma\gamma(\overline{g} - \underline{g} - 2\delta)}{\Gamma} \end{aligned} \tag{42}$$

which contradicts (41) if ϵ is small enough. This concludes the proof of convergence of $g(B(t))$. □

Remark G.4 *Since $g(B)$ is a strictly increasing function of B, $B(t)$ converges by Lemma G.5.*

Lemma G.6 *Suppose for some t_0 that $g(B(t)) \in [a, b]$ for all $t \geq t_0$. For any i, if $U_i'(x_i(t_1)) \in [a, b]$ for some $t_1 \geq t_0$, then $U_i'(x_i(t)) \in [a, b]$ for all $t > t_1$.*

Proof. The proof is based on the mean value theorem. Suppose there is a $t > t_1$ such that $U_i'(x_i(t)) \notin [a, b]$. Without loss of generality, let $U_i'(x_i(t)) > b$. Define $t_2 = \sup\{t : U_i'(x_i(t)) \leq b,\ t \geq t_1\}$. There exists $t_2 < \tilde{t} < t$ such that $\frac{d}{dt}U_i'(x_i(\tilde{t})) > 0$, which contradicts the fact that $U_i'(x_i(\tilde{t})) > b \geq g(B(\tilde{t}))$. □

With the lemmas derived above, we complete the proof of Proposition 4.1.

Proof of Proposition 4.1 By Remark G.4, suppose $B_\infty = \lim_{t\to+\infty} B(t)$. Let $\mathcal{U}_+ = \{i : U_i'(0) \geq g(B_\infty)\}$ and $\mathcal{U}_- = \{i : U_i'(0) < g(B_\infty)\}$. Define

$$p_o = \begin{cases} \max_{i\in\mathcal{U}_-} U_i'(0), & \text{if } \mathcal{U}_- \neq \phi, \\ -\infty, & \text{else.} \end{cases}$$

For any small ϵ satisfying $g(B_\infty - \epsilon) > p_o$, there exists T_ϵ such that $B(t) \in [B_\infty - \epsilon, B_\infty + \epsilon]$ for all $t \geq T_\epsilon$. Define

$$\begin{aligned} \mathcal{G}_+^\epsilon(t) &= \{i : U_i'(x_i(t)) > g(B_\infty + \epsilon)\} \\ \mathcal{G}_-^\epsilon(t) &= \{i : U_i'(x_i(t)) < g(B_\infty - \epsilon)\} \\ \mathcal{G}_0^\epsilon(t) &= \{i : g(B_\infty - \epsilon) \leq U_i'(x_i(t)) \leq g(B_\infty + \epsilon)\} \end{aligned}$$

Clearly, $\mathcal{U}_- \subseteq \mathcal{G}_-^\epsilon(T_\epsilon)$. Consider the buyers in four disjoint sets: $\mathcal{G}_+^\epsilon(T_\epsilon)$, $\mathcal{G}_0^\epsilon(T_\epsilon)$, $\mathcal{G}_-^\epsilon(T_\epsilon)\backslash\mathcal{U}_-$ and $\mathcal{U}_-$.

1. If $i \in \mathcal{G}_+^\epsilon(T_\epsilon)$, suppose $i \in \mathcal{G}_+^\epsilon(t)$ over $[T_\epsilon, +\infty)$, $b_i(t)$ goes to $+\infty$ asymptotically since $\dot{b}_i(t) = b_i(t)$. Since $B(t)$ is bounded, $U_i'(x_i(t))$ will enter the interval $[g(B_\infty - \epsilon), g(B_\infty + \epsilon)]$ in finite time.

2. If $i \in \mathcal{G}_0^\epsilon(T_\epsilon)$, $U_i'(x_i(t)$ is in the interval $[g(B_\infty - \epsilon), g(B_\infty + \epsilon)]$ at time T_ϵ.

3. If $i \in \mathcal{G}_-^\epsilon(T_\epsilon) \backslash \mathcal{U}_-$, suppose $i \in \mathcal{G}_-^\epsilon(t)$ over $[T_\epsilon, +\infty)$, $b_i(t)$ goes to 0 asymptotically since $\dot{b}_i(t) = -b_i(t)$. Then $x_i(t) = \frac{b_i(t)}{B(t)} C$ also goes to 0 asymptotically since $B(t)$ is bounded. Hence, $U_i'(x_i(t))$ will hit the interval $[g(B_\infty - \epsilon), g(B_\infty + \epsilon)]$ in finite time because $U_i'(0) > g(B_\infty - \epsilon)$.

4. If $i \in \mathcal{U}_-$, compare to the case above, $U_i'(x_i(t))$ can never hit interval $[g(B_\infty - \epsilon), g(B_\infty + \epsilon)]$ since $U_i'(0) < g(B_\infty - \epsilon)$. Hence, $x_i(t)$ goes to 0 asymptotically.

To summarize the conclusions in the four cases, if $i \in \mathcal{G}_+^\epsilon(T_\epsilon) \bigcup \mathcal{G}_0^\epsilon(T_\epsilon) \bigcup (\mathcal{G}_-^\epsilon(T_\epsilon) \backslash \mathcal{U}_-) = \mathcal{U}_+$, $U_i'(x_i(t))$ will enter the interval $[g(B_\infty - \epsilon), g(B_\infty + \epsilon)]$ in finite time. After that, $U_i'(x_i(t))$ will always stay in the interval $[g(B_\infty - \epsilon), g(B_\infty + \epsilon)]$ by Lemma G.6. Since ϵ can be any small number, $\lim_{t \to +\infty} U_i'(x_i(t)) = g(B_\infty)$ for all $i \in \mathcal{U}_+$. Also, $\lim_{t \to +\infty} U_i'(x_i(t)) = U_i'(0)$ for all $i \in \mathcal{U}_-$.

Thus, $\boldsymbol{x}(t)$ and $\boldsymbol{b}(t)$ converge. Let $\boldsymbol{b}(t)$ converge to $\boldsymbol{b}_\infty$. If $i \in \mathcal{U}_-$, then $x_i(\boldsymbol{b}_\infty) = 0$ and $U_i'(x_i(b_\infty)) < g(B_\infty)$. If $i \notin \mathcal{U}_-$, then $U_i'(x_i(\boldsymbol{b}_\infty)) = g(B_\infty)$. So with $\lambda = g(B_\infty)$, (2) is satisfied by λ and $\boldsymbol{x}(\boldsymbol{b}_\infty)$. So $\boldsymbol{x}(\boldsymbol{b}_\infty)$ is the efficient allocation. Since the efficient allocation is unique, $\boldsymbol{b}_\infty$ is unique. □

H Proofs of the results for mechanisms PQ2 and PQ1

In this appendix, the results of Section 5 are proved.

Proof of Proposition 5.1 Suppose $(x_1^*, \ldots, x_n^*)$ is the efficient allocation. Let $\boldsymbol{\sigma^*}$ be given by $w_i^* = c x_i^*$ for all i and $a^* = b^* = \lambda$, where c is an arbitrary positive constant. Note that $\overline{p}(\boldsymbol{\sigma^*}) = \lambda$, and no single buyer can cause $\overline{p}$ to decrease. Consequently $\boldsymbol{\sigma^*}$ is an NEP, and the allocation under $\boldsymbol{\sigma^*}$ is efficient.

It remains to show that there are no other NEPs. Suppose $\boldsymbol{\sigma} = (w_1, \ldots, w_{n-1}, a, w_n, b)$ is an NEP, with corresponding allocation vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ and price $\overline{p}$. Note that $\overline{p}$ is determined by (a, b), whereas $(x_1, \ldots, x_n)$ is determined by $(w_1, \ldots, w_n)$. By the assumptions on the valuation functions, $x_i > 0$ for all i. It must be impossible for buyer $n-1$ or buyer n alone to cause $\overline{p}$ to decrease, so that $a = b$. Therefore, no buyer can cause the price to decrease. Thus, for each buyer i, w_i must be selected to maximize $U_i(x_i) - \overline{p} x_i$ (for the other bids fixed), so that $x_i = {U'}_i^{-1}(\overline{p})$. Since $\sum_{i=1}^n x_i = C$, it follows therefore that $\sum_{i=1}^n {U'}_i^{-1}(\overline{p}) = C$, which means that $a = b = \lambda$. Thus, $\boldsymbol{\sigma} = \boldsymbol{\sigma^*}$ for some choice of c. □

Proof of Proposition 5.2 Suppose $(x_1^*, \ldots, x_n^*)$ is the efficient allocation and λ is the market clearing price for price taking buyers. Let $\boldsymbol{\sigma^*} = (w_1^*, \ldots, w_n^*, p^*)$ be given by $w_i^* = \lambda x_i^*/(C - x_n^*)$ for all i and $p^* = \lambda$ for all i. Note that $p_i(\boldsymbol{\sigma^*}) = \lambda$ for all i, $x_i(\boldsymbol{\sigma}) = x_i^*$ for all i, and no single buyer can cause her own price to decrease. Consequently $\boldsymbol{\sigma^*}$ is an NEP, and the allocation under $\boldsymbol{\sigma^*}$ is efficient.

It remains to show that $\boldsymbol{\sigma^*}$ is the unique NEP. Suppose $\boldsymbol{\sigma} = (w_1, \ldots, w_n, p)$ is an NEP, with corresponding allocation vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ and price vector $\boldsymbol{p} = (p_1, \ldots, p_n)$. Clearly $p = \sum_{j=1}^{n-1} w_j$, for otherwise buyer n could decrease her own price. Thus, all buyers pay the same average price, p. For each i with $1 \leq i \leq n-1$, buyer i cannot influence her price, and buyer n cannot decrease her price. Each buyer, for the bids of other buyers fixed, could obtain any quantity in $(0, C)$. Thus, it must be that w_i causes x_i to maximize $U_i(x_i) - px_i$, or equivalently, $x_i = {U'}_i^{-1}(p)$. Since $\sum_{i=1}^n x_i = C$, it follows therefore that $\sum_{i=1}^n {U'}_i^{-1}(p) = C$, which means that $p = \lambda$. Thus, $\boldsymbol{\sigma} = \boldsymbol{\sigma^*}$. □

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