Comments

Comments on "An Optimal Shortest-Path Routing Policy for Network Computers with Regular Mesh-Connected Topologies"

Timothy Weller and Bruce Hajek

***Abstract*— Badr and Podar introduced a zig-zag routing policy and showed its optimality for *shortest-path routing* on square or infinite grid networks with independent link failures. These comments show that, contrary to the claim of Badr and Podar, a zig-zag policy is not optimal for shortest-path routing on torus networks.**

***Index Terms*—Dynamic programming, dynamic routing, grid networks, mesh-connected topologies, shortest-path routing.**

I. Introduction

Consider the packet routing problem on the three types of $N \times N$ two-dimensional grid networks shown in Fig. 1. Note that N is infinite for the infinite grid, and that the torus has "wraparound" paths at the boundaries. During a given time slot, some of the outgoing links at a node may be unavailable due, for example, to a physical link failure or competing traffic. However, as long as a packet is in the network, it must traverse a link in each time slot. Under *shortest-path routing*, only shortest paths are acceptable. If a packet cannot continue along a shortest path to its destination, it is discarded.

Suppose there is a packet at a node (i, j) destined for node $(0, 0)$. Let $N_{\mathrm{SP}}(i, j)$ be the set of neighboring nodes of (i, j) which lie along a shortest path to the destination. Shortest path routing requires the packet to move to a node in $N_{\mathrm{SP}}(i, j)$. The packet expresses its preferences by choosing an ordering of the acceptable outgoing links (i.e., those links going to nodes in $N_{\mathrm{SP}}(i, j)$). This ordering of acceptable links is called the *control*, $\mu(i, j)$, where the kth preference is $\mu^k(i, j)$. A function μ mapping each (i, j) to a control $\mu(i, j)$ is called a *routing policy.* The problem is to find an *optimal routing policy.* Such a policy should minimize the expected delivery time of the packet to the destination. The stochastic aspect of the model is the fact that after choosing its ordering of acceptable links, the packet actually departs to node $\mu^k(i, j)$ with some probability $pk(i, j)$.

Assuming links fail independently with probability $1 - p$, it is clear that $pk(i, j) = pk = p(1 - p)^{k-1}, 1 \leq k \leq |N_{\mathrm{SP}}(i, j)|$. Note that unlike $N_{/rmSP}$ for the square or infinite grid, $N_{/rmSP}(i, j)$ for the torus can contain more than two nodes for some nodes (i, j), due to the wraparound. A *zig-zag routing policy* (Z^2*policy*) is one in which the packet always prefers to move toward the nearest *diagonal* $(i = \pm j)$. Badr and Podar [1] showed that a Z^2 policy is optimal for the square and infinite grid networks. They claimed [1, cf. first paragraph of Section IV] that a Z^2 policy is also optimal for torus networks, but the torus defies initial intuition as demonstrated by the following fact.

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T. Weller was with Coordinated Science Laboratory and Department of Electrical and Computer Engineering, University of Illinois, Urbana, IL 61801 USA. He is now with Donaldson, Lufkin & Jenrette, 35th Floor, 140 Broadway, New York, NY 10005 USA.

B. Hajek is with Coordinated Science Laboratory and Department of Electrical and Computer Engineering, University of Illinois, Urbana IL 61801 USA.

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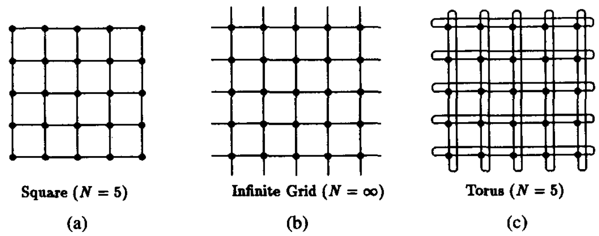

Fig. 1. Some types of two-dimensional grid networks.

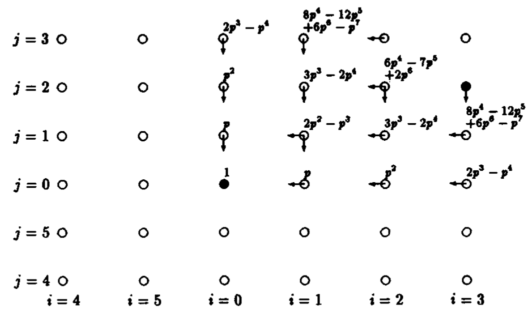

Fig. 2. $P^*(i, j)$ and preferred outgoing links for a 6×6 torus.

Fact: Consider a packet subject to shortest-path routing on a torus network with independent link failures. A Z^2 policy does not necessarily maximize the probability of delivery of the packet to the destination. In particular, a Z^2 policy is not optimal on a 6×6 torus.

Proof: Define $P^*(i, j)$ as the maximum probability of delivery of a packet at node (i, j) to $(0, 0)$, the maximum being over all routing policies. Then P^* satisfies the equations

$$P^*(i, j) = \max_u \sum_{k=1}^{|N_{SP}(i,j)|} p_k P^*\left(u^k\right)$$
$$P^*(0, 0) = 1, \quad (1)$$

where $u = (u^1, u^2, u^3, u^4)$ ranges over the permutations of the elements of $N_{\mathrm{SP}}(i, j)$. These equations can be derived by conditioning on the first move from node (i, j), and they are called the dynamic programming (DP) equations [2]. A function mapping each (i, j) into a control that achieves the maximum above is an optimal routing policy. The DP equations can be easily solved to get $P^*(i, j)$ for each (i, j) by working away from the destination. Some values are shown in Fig. 2 for $N = 6$. The arrows indicate the first preferred direction from each node (no preference on either diagonal). Note that the packet first attempts to go to the neighboring node with highest P^* value since this equals the probability of delivery which is to be maximized.

As indicated in Fig. 2, the first preference of a packet at node (3,2) is to go to node (3,1) and not (2,2), since $P^*(3, 1) - P^*(2, 2) = p^4(1 - p)^2(2 - p) > 0$ for $0 < p < 1$. Thus, a Z^2 policy is not optimal for shortest-path routing on a 6×6 torus network with independent link failures. For a larger $N \times N$ torus, P^* can be computed numerically, and again the same pattern emerges. □

The optimal policy for the torus seems unlikely to be of a simple closed form, and it can be seen that Z^2 routing is very close to optimal by comparing P^* and the actual packet delivery probabilities under a Z^2 routing policy. Finally, the results of [1] and these comments were extended to *deflection routing* on a torus and an infinite grid [3].

References

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