On the Capture Probability for a Large Number of Stations

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***Abstract*—The probability of capture under a model based on the ratio of the largest received power to the sum of interference powers is examined in the limit of a large number of transmitting stations. It is shown in great generality that the limit depends only on the capture ratio threshold and the roll-off exponent of the distribution of power received from a typical station. This exponent is insensitive to many typical channel effects such as Rician or Rayleigh fading and log-normal shadowing. The model is suitable for large systems with noncoherently combined interference.**

***Index Terms*— Direct-sequence spread-spectrum, log-normal shadowing, near–far effect.**

I. Introduction

CONSIDER THE following power capture model. If n transmitters are present, the transmission of a given station j is captured if

$$P_{R,j} \geq z\left\{\sum_{i\neq j} P_{R,i} + N\right\} \tag{1}$$

where z is the power ratio threshold, $P_{R,i}$ is the received power at the base station due to transmitter i, and N is a nonnegative random variable that represents the effect of additive noise, such as receiver noise or interference from transmitters in other systems. In a land mobile radio system, the received power P_R from a remote station at radius r can be reasonably modeled as

$$P_R = R^2 K_s e^{\xi} K r^{-\beta} P_T \tag{2}$$

where R is Rician or Rayleigh distributed, ξ is Gaussian distributed, with zero mean and standard deviation σ_s, $K_s = \exp(-\sigma_s^2/2)$ (so that $E[K_s \exp(\xi)] = 1$), and P_T is the transmitted power. The term R accounts for diffuse multipath fading with or without a specular component (i.e., Rician or Rayleigh fading), the log-normal term $K_s e^{\xi}$ models the effects of shadowing, and the term $Kr^{-\beta}$ reflects the power attenuation with distance. Although the above propagation model is based on observations in the land mobile environment, fading and shadowing have also been observed in the indoor environment [1]. Depending on the characteristics of the environment, β has been observed to be in the range from two for free-space to nearly six for cluttered paths [1]. The propagation model described by (2) is *multiplicative* in that the received power is obtained by multiplying the transmitted power by some random variables. Several other propagation models, such as those involving the Nakagami distribution [2, p. 40], are also multiplicative. The asymptotic analysis given in this paper applies to a wide range of multiplicative propagation models, not just the model specifically detailed in (2).

The parameter z is the minimum carrier-to-interference ratio (CIR) needed for successful reception, and is determined by such factors as the type of modulation and the receiver sensitivity. For typical narrowband systems, z is in the range $1 < z < 10$. For a direct-sequence spread-spectrum (DS/SS) system, the processing gain effectively reduces the effect of interference from other transmitters, so the value of z is roughly inversely proportional to the processing gain. For such systems, z in the range $0.1 < z < 1$ is typical. It is assumed here only that $z > 0$. If $z \leq 1$, then it is possible that more than one signal is captured. In general, at most $1 + \lfloor 1/z \rfloor$ transmissions can be captured.

If a *snapshot* of a system is to be modeled, the distance r of a typical station from the base is assumed to be random with some distribution function F_R. This makes the term $Kr^{-\beta}$ random. The term models the near–far effect, and its probability distribution is determined by the spatial distribution of stations and the relationship between power attenuation and distance. We invoke the simplifying assumption that the fading, shadowing, and locations of distinct stations, and thus the received power for distinct stations, are mutually independent. Recall that P_R denotes the (random) received power from a remote station, and let $F_P^c(x) = P[P_R > x]$ denote the complementary distribution function of P_R. If there is a constant c_P so that $F_P^c(x) \sim c_P x^{-1/\delta}$ (which by definition means that $F_P^c(x)x^{1/\delta} \to c_P$) as $x \to \infty$, then δ is called the roll-off parameter of the distribution of power received from a typical station. It is discussed further in the next section.

The purpose of this paper is to make two points.

1) Under broad conditions, the roll-off parameter δ of the distribution of power received from a typical station is

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determined by β and F_R through the near–far effect. The parameter δ is insensitive to other effects such as Rayleigh or Rician fading and log-normal shadowing.

2) In the limit of a large number of transmitters, the probability of capture is determined by z and the roll-off parameter δ.

These two points are addressed in Sections II and III, respectively. A numerical example is given in Section IV and final remarks are presented in Section V.

A comment on the basic model of the paper is in order here. The basic result is a limit theorem, in which the number of stations tends to infinity. In engineering practice, limit theorems justify, or at least suggest, approximations. In the case at hand, the results of this paper suggest an approximate capture probability for a base station surrounded by a large number of stations. In practice, one could think of a fixed cell, and then let the number of stations within the cell become very large. Typically, transmissions from only the stations closest to the base station are captured. Reasonably speaking, one would not expect stations to get arbitrarily close to the base station, which is implied by letting the number of stations become large. However, we have found that the limiting probability of capture is approached closely even with a small number of transmitting stations (see Fig. 1).

There is another scenario with a large number of stations. Another, interesting and realistic limit scenario is now considered. Suppose that there is a base station, which we designate base station zero, serving stations within a cell of approximate fixed radius R_0. Suppose further that the cell is one of many cells in a much larger region, with radius R_1. One can increase the number of transmitting stations by increasing R_1, with the density of transmitters per unit area fixed. Note that the transmitting stations far from base station zero are not even attempting to be captured by base station zero, since they are closer to other base stations. Nevertheless, they contribute to the interference at base station zero. The results of this paper nearly apply to this scenario, the exception being the proper treatment of the additive noise in this case.

In previous work, it has been shown for specific examples that the limit of the probability of capture is insensitive to Rayleigh fading, shadowing, and the characteristics of the spatial distribution of stations outside a neighborhood of the base station [3], [4]. The past results on computing the limit of the probability of capture all use one or both of the following two assumptions: 1) Rayleigh fading is present, which allows the random variable R^2 in (2) to have an exponential distribution and hence allows for an analytical treatment [3]–[5], or 2) the spatial distribution of the stations follows the quasi-uniform distribution [4], [5] given by the density function $f(r) = 2re^{-\pi r^4/4}$, $r \geq 0$, where r is the distance from the base to the station. This quasi-uniform distribution permits one to analytically compute the distribution of the summed power of a number of stations. However, the sensitivity of these past results to the exact assumptions used is unclear and it is desirable to know if the results would still hold if the assumptions were violated. The results presented in this paper show the insensitivity of the limit of the probability of capture and encompass all of the above-mentioned previous observations as special cases.

II. Dominance of the Near–Far Effect on Probability Roll-Off

To introduce ideas, we begin our discussion with a simple case. Suppose the only factor governing the received power differentials is the near–far effect, so that $P_R = Kr^{-\beta}P_T$, where K and P_T are constant and the same for all transmitters. Furthermore, suppose that the stations are uniformly distributed within the unit disk, so the probability a station is within distance r of the base is $F_R(r) = r^2$ for $0 \leq r \leq 1$. Then

$$F_P^c(x) = P[P_R > x] = P[P_T K r^{-\beta} > x] \\ = F_R[(x/P_T K)^{-1/\beta}] = (x/P_T K)^{-2/\beta}. \quad (3)$$

Thus, $F_P^c(x) \sim c_P x^{-2/\beta}$ for some constant c_P. More generally, if the distribution function of distance from the base station satisfies $F_R(r)r^{-a} \rightarrow c_R$ as $r \rightarrow 0$, then $F_P^c(x) \sim c_P(x^{-1/\beta})^a = c_P x^{-a/\beta}$ as $x \rightarrow \infty$. In this case, the distribution of the power received from a typical station has roll-off exponent $\delta = a/\beta$.

If the spatial distribution is "punctured," meaning there is a positive lower bound on how close stations can be to the base, then $F_P^c(x) = 0$ for x sufficiently large, which formally corresponds to $\delta = \infty$.

For the example above, the tail probability $F_P^c(x)$ falls off as a negative power of x as $x \rightarrow \infty$. In contrast, the tail probability falls off much more quickly for the random variables R, e^{ξ}, and other variables commonly used to model channel propagation characteristics. According to the following proposition, these other variables do not affect the negative exponent of x describing the falloff of the tail distribution. The proof of the proposition is given in Appendix A.

Proposition 2.1: Let $Y = XT$, where X and T denote independent, nonnegative random variables. Suppose for some $\delta > 0$, $0 < c_X < \infty$ and $\eta > 0$, that $F_X^c(x) \sim c_X x^{-\delta}$ as $x \rightarrow \infty$, $\lim_{t\rightarrow\infty} F_T^c(t)t^{\delta+\eta} = 0$, and $P[T > 0] > 0$. Then $F_Y^c(y) \sim c_Y y^{-\delta}$ as $y \rightarrow \infty$, for some constant $0 < c_Y < \infty$.

For example, suppose that X represents the received power from a typical station if, as above, only the near–far effect is taken into account. Then $\delta = a/\beta$. The random variable T can be taken to be a Rayleigh, Rician, log-normal, Suzuki, or Nakagami distributed random variable [2], for example, since the tails of all these distributions fall off at least exponentially fast, easily insuring that the condition on F_T^c required in the proposition is satisfied.

For these cases, $P[XT \geq x] \sim cx^{-a/\beta}$. Repeated application of the proposition implies that $P[XT_1T_2\cdots T_k \geq x] \sim cx^{-a/\beta}$, where $T_1, \cdots, T_k$ are independent random variables, and each is one of the types listed above. That is, $XT_1T_2\cdots T_k$ has the same roll-off exponent as X alone.

III. Insensitivity of Capture Probability for a Large Number of Stations

The proposition below makes the point that in the limit of a large number of stations, the capture probability tends to a limit determined by z and the roll-off exponent δ of the tail of the distribution of received power from a typical station. Suppose the received powers for n stations are independent and identically distributed with common distribution function F, and that the additive noise term N is independent of the received powers and has a distribution that does not depend on n. Let $P(n,k,z,F)$ denote the probability that at least k transmissions are captured, and let $\Phi(n,z,F)$ denote the expected number of transmissions that are captured. Note that $P(n,k,z,F)=0$ for $k>1+\lfloor 1/z\rfloor$, and that $\Phi(n,z,F)=\sum_{k=1}^{\infty}P(n,k,z,F)$ (since if X is a nonnegative integer-valued random variable, then $E[X]=\sum_{k=1}^{\infty}P[X\geq k]$). If $z>1$, then at most one transmission can be captured, and $P(n,1,z,F)$ and $\Phi(n,z,F)$ are both equal to the probability that capture occurs.

The asymptotic result is described using the following artificial infinite system. Imagine that the base station is located at the origin of the real line, and there are remote stations located at the points $S_1, S_2, \cdots$ of a Poisson point process of unit intensity on the positive half line. Thus, there is a sequence of independent, exponentially distributed random variables with mean one, and S_j is the sum of the first j variables of the sequence. Imagine also that a station at coordinate s generates received power $s^{-1/\delta}$ at the base station. Equivalently, the received power from a station is at least x if and only if the station is in the interval $[0, x^{-\delta}]$. Given that a station lies in $[0,L]$ for some large constant L, its location is uniformly distributed over $[0,L]$ by the nature of Poisson processes. For such a station, the conditional distribution of received power has complementary distribution function $F^c(x)=(x^{-\delta})/L$ for sufficiently large x, and this distribution has roll-off parameter δ. Thus, intuitively, the artificial infinite system is closely related to large finite systems with the same δ. Let $p(k,z,\delta)$ denote the probability that at least k transmissions are captured, and let $\phi(z,\delta)$ denote the expected number of transmissions captured, for the infinite system.

Proposition 3.1: Suppose $z>0$. If for some $\delta>0$, $F^c(x)\sim cx^{-\delta}$ for some nonzero finite constant c, then

$$\lim_{n\to\infty}P(n,k,z,F)=p(k,z,\delta), \qquad \text{for } k\geq 1 \tag{4}$$

$$\lim_{n\to\infty}\Phi(n,z,F)=\phi(z,\delta). \tag{5}$$

Furthermore,

$$\phi(z,\delta)=\begin{cases} z^{-\delta}\,\mathrm{sinc}(\delta), & \text{if } 0<\delta<1\\ 0, & \text{if } \delta\geq 1\end{cases} \tag{6}$$

Here $\mathrm{sinc}(\delta)=\sin(\pi\delta)/(\pi\delta)$. Proposition 3.1 is proved in Appendixes B and C.

IV. Implications and Numerical Results

Together, Propositions 2.1 and 3.1 imply that for a large class of spatial distributions, Rayleigh and Rician fading and log-normal shadowing do not affect the large n limit of the capture probability. For example, this class includes all spatial distributions $F_R(r)$ for which $F_R(r)r^{-a}\to c_R$ as $r\to 0$. For the uniform and quasi-uniform cases that are treated in [3] and [4], respectively, $a=2$, and the large n limit of the capture probability given by (5) becomes $z^{-\frac{2}{\beta}}\mathrm{sinc}(\frac{2}{\beta})$. This result is consistent with the result of [3] for a case including the near–far effect, Rayleigh fading, and log-normal shadowing. Furthermore, for the more specific case of $\beta=4$, (5) becomes simply $\frac{2}{\pi\sqrt{z}}$, which is consistent with the result of [2, p. 256] for a case with the near-far effect and Rayleigh fading and with the result of [4] for the near–far effect only case.

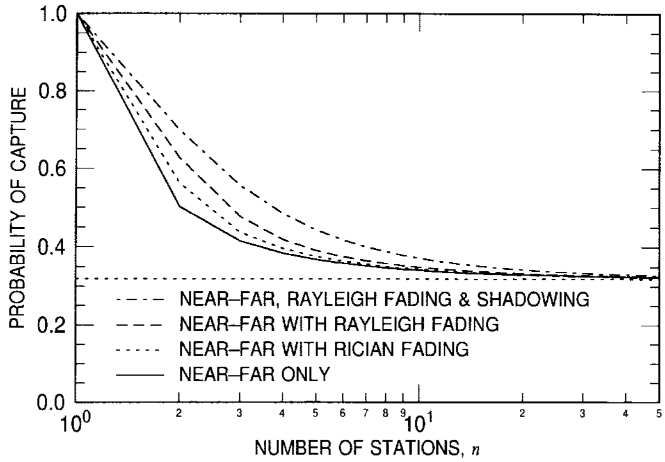

Fig. 1. Capture probabilities for $z=6$ dB, $\beta=4$, $s_s=6$ dB, and $K_{\mathrm{Rician}}=6$ dB.

We consider a numerical example for a uniform spatial distribution of remote stations, a power ratio threshold z of 6 dB (or 3.98), and a roll-off exponent β of four. These parameters yield a capture probability limit of $\phi(3.98, 0.5)=0.319$ as determined by (5), where $\delta=2/\beta$ for this case. In Fig. 1, we show the capture probabilities for cases in which the near–far effect alone is present and for cases that also include Rician fading with a line-of-sight (LOS) to multipath signal-power ratio $K_{\mathrm{Rician}}=6$ dB (see [1] or [2]) and Rayleigh fading. The case of combined Rayleigh and log-normal shadowing is also shown (i.e., the Suzuki distribution, see [2]). For this shadowing case, we note that in (2) the standard deviation of e^{ξ} expressed in decibels (i.e., the standard deviation of $10\log_{10}(e^{\xi})$) is denoted by s_s and is related to σ_s by $\sigma_s=0.1\ln(10)s_s$. We have chosen a shadowing parameter s_s of 6 dB in Fig. 1. All of these fading and shadowing cases have the same capture probability limit as the case that has the near–far effect only. Both of the Rayleigh fading cases were computed using numerical integration (see [3] and [6]). The remaining two cases were generated using very long simulations in which the resulting confidence intervals were too small to show.

V. Final Comments

Caution should be used not to place too much importance on the large n limit of capture probability. First of all, as emphasized in [3], it is not necessarily an upper bound on the achievable throughput, because even if n is large, a retransmission control strategy can be used to effectively decrease the number of simultaneous transmitters to a near-optimal level. Second of all, when n is large, the stations that are successful will tend to be closer to the base station, so that after some period of operation relatively fewer of the closer stations will require transmission. This could severely decrease the actual observed capture rates in practice.

Nevertheless, the identification of the large n limit is useful, since: 1) it offers a convenient check on numerical computations for finite n, 2) for large, highly mobile networks, it offers a simple approximation for the expected capture rate.

A related research problem that is worthy of further investigation is the determination of the large n limit of the capture probability for the case of a diversity system (e.g., multiple receiving antennas). This problem was investigated for the case of a dual-diversity system with independent Rayleigh fading by Zorzi [7]. Given some assumptions on the spatial distribution, Zorzi's results indicate that the large n limit does in fact depend on the presence of Rayleigh fading unlike the results for the no-diversity case studied in this paper.

An interesting extension of this work would lie in finding the appropriate density of base stations to serve a population of mobile users, along the lines of the spectrum efficiency described in [8]. The straightforward application of the results in this paper apply to the case of one base station with a large population of mobile users. However, one could imagine a large population of mobile users together with a number of base stations located at different spatial locations. In that case, our results could apply to the effective throughput seen at each base station, though some additional work will be required to determine joint statistics. These throughputs together with the desired system capacity can be used to determine the correct (minimum) density of base stations to achieve a given level of performance.

The basic model studied in this paper assumes that transmissions are not power controlled. In some large cellular systems, stations may exert power control, but many of the stations may be controlling power received at base stations other than at base station zero (the particular base station of interest, as in the introduction). Thus, the power received at base station zero from each of a large number of distant interfering stations is not tightly controlled. Perhaps the results of this paper can be modified to provide insight into the effects of such interference, in particular in connection with advanced methods of multiple-access detection at base station zero.

Appendix A
Proof of Proposition 2.1

Given $\epsilon > 0$, choose $\alpha > 0$ so small that $|F_X^c(x)x^\delta - c_X| \leq \epsilon$ for $x > 1/\alpha$. This inequality and the fact $F_X^c(x) \leq 1$ imply

$$\begin{aligned} F_Y^c(y) &= \int_{0+}^{\infty} F_X^c\Big(\frac{y}{t}\Big) dF_T(t) \\ &= \int_{0+}^{\alpha y} F_X^c\Big(\frac{y}{t}\Big) dF_T(t) + \int_{\alpha y+}^{\infty} F_X^c\Big(\frac{y}{t}\Big) dF_T(t) \\ &\leq (c_X + \epsilon) y^{-\delta} \int_{0+}^{\alpha y} t^\delta dF_T(t) + F_T^c(\alpha y) \\ &\leq (c_X + \epsilon) y^{-\delta} E[T^\delta] + o((\alpha y)^{-\delta}). \end{aligned}$$

Furthermore, integration by parts and the assumptions on F_T yield that

$$\begin{aligned} E[T^\delta] = \int_0^1 t^\delta dF_T(t) - t^\delta F_T^c(t)|_1^\infty \\ + \delta \int_1^\infty t^{\delta-1} F_T^c(t) dt < \infty. \end{aligned}$$

Similarly

$$\begin{aligned} F_Y^c(y) &\geq (c_X - \epsilon) y^{-\delta} \int_{0+}^{\alpha y} t^\delta dF_T(t). \\ &\geq (c_X - \epsilon) y^{-\delta} (E[T^\delta] - \epsilon) \text{ all large } y. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $\lim_{y \to \infty} F_Y^c(y) y^\delta = c_X E[T^\delta]$. Proposition 2.1 is proved.

Appendix B
Universality of the Limit

The proof of Proposition 3.1 is divided between this and the next Appendix. Equation (4) is verified in this Appendix. Equation (5) follows immediately from (4) since at most $1 + \lfloor 1/z \rfloor$ transmissions can be captured for either the finite n or the infinite n model. The next Appendix completes the proof by establishing (5). Fix k with $k \geq 1$ for the remainder of this Appendix.

We pause to give some intuition behind the proof of (4). The distribution of power received from a typical station has the same roll-off parameter as the random variable $U^{-1/\delta}$, where U is uniformly distributed on $[0, 1]$. The kth largest power received from n independent stations is thus distributed like $(U^{(k)})^{-1/\delta}$, where $U^{(k)}$ is the kth smallest of n independent random variables, each uniformly distributed on $[0, 1]$. By the well-known theory of order statistics (see [9, Section I.7] or [10, p. 335]) the joint distribution of $(nU^{(1)}, \cdots, nU^{(J)})$ converges as $n \to \infty$ to the distribution of $(S_1, \cdots, S_J)$. Thus, for large n and J

$$\begin{aligned} P(n,k,z,F) &\approx P\Big[(U^{(k)})^{-1/\delta} \geq \frac{z}{1+z}\big\{(U^{(1)})^{-1/\delta} + \cdots \\ &\qquad + (U^{(J)})^{-1/\delta} + N\big\}\Big] \\ &= P\Big[(nU^{(k)})^{-1/\delta} \geq \frac{z}{1+z}\big\{(nU^{(1)})^{-1/\delta} + \cdots \\ &\qquad + (nU^{(J)})^{-1/\delta} + n^{-1/\delta} N\big\}\Big] \\ &\approx P\Big[S_k^{-1/\delta} \geq \frac{z}{1+z}\big\{S_1^{-1/\delta} + \cdots + S_J^{-1/\delta}\big\}\Big] \\ &\approx p(k,z,\delta). \end{aligned}$$

We now turn to the actual proof of (4).

Let F denote a distribution function with $F(0-) = 0$. Given $n \geq 1$, let $Z_1, \cdots, Z_n$ be independent, identically distributed random variables with distribution function F. Let $Z^{(1)}, \cdots, Z^{(n)}$ denote the corresponding *decreasing* sequence of order statistics, which by definition is formed by a random permutation of $Z_1, \cdots, Z_n$, such that $Z^{(1)} \geq \cdots \geq Z^{(n)}$. Then $P(n,k,z,F)$ can be expressed as $P(n,k,z,F) = P[Z^{(k)} \geq \frac{z}{1+z}\{Z^{(1)} + \cdots + Z^{(n)} + N\}]$. Assume that $F^c(x) \sim cx^{-\delta}$ as $x \to \infty$, where $\delta > 0$ and $0 < c < \infty$.

Similarly, $p(k,z,\delta)$ can be expressed as

$$p(k,z,\delta) = P\left[S_k^{-1/\delta} \geq \frac{z}{1+z}\{S_1^{-1/\delta} + S_2^{-1/\delta} + \cdots\}\right]. \tag{6}$$

The law of large numbers implies that $S_j/j \to 1$ with probability one as $j \to \infty$, which implies that $p(k,z,\delta) = 0$ for $\delta \geq 1$.

Lemma 7.1: If $F^c(x) \sim cx^{-\delta}$, where $\delta > 0$ and $0 < c < \infty$, then there is a bounded function a on the interval (0, 1) with $\lim_{u \to 0} a(u) = 1$ such that if U is uniformly distributed on the unit interval, then $[U/a(U)c]^{-1/\delta}$ is monotone nonincreasing in U and has distribution function F.

Proof: Following standard theory, define g by $g(u) = \inf\{x: F^c(x) \leq u\}$ for $0 < u < 1$, so that g is a version of the inverse function of F^c. Then g is nonincreasing, right continuous, and $g(u) \leq x$ if and only if $u \geq F^c(x)$. This last relation implies that $g(U)$ has distribution function F. Set $a(u) = u[g(u)]^\delta/c$ so that $g(U) = [U/a(U)c]^{-1/\delta}$. In view of the properties of g, it remains only to show that $\lim_{u\to 0} a(u) = 1$. This last relation follows from the following fact, which itself is easy to verify by a simple sketch: If for $\epsilon > 0$ and $x_o > 0$, $|F^c(x)x^\delta - c| \leq \epsilon$ for $x \geq x_o$, then

$$|a(u) - 1| \leq \epsilon/c \quad \text{for } 0 < u \leq cx_o^{-\delta}(1-\epsilon).$$

□

In view of the lemma, we can write for $1 \leq J \leq n$

$$\begin{aligned} &P(n,k,z,F) \\ &= P\left[\left[\frac{nU^{(k)}}{a(U^{(k)})}\right]^{-1/\delta} \geq \frac{z}{1+z}\left\{\left[\frac{nU^{(1)}}{a(U^{(1)})}\right]^{-1/\delta} + \cdots \right.\right. \\ &\qquad \left.\left. + \left[\frac{nU^{(J)}}{a(U^{(J)})}\right]^{-1/\delta} + R + n^{-1/\delta}N\right\}\right] \end{aligned} \tag{7}$$

where $U^{(1)}, \cdots, U^{(n)}$ denotes the *increasing* sequence of order statistics of n independent, uniformly distributed random variables, and

$$R = \left[\frac{nU^{(J+1)}}{a(U^{(J+1)})}\right]^{-1/\delta} + \cdots + \left[\frac{nU^{(n)}}{a(U^{(n)})}\right]^{-1/\delta}.$$

The representation (7) is useful since, as well known from the theory of order statistics, $(nU^{(1)}, \cdots, nU^{(J)}) \Rightarrow (S_1, \cdots, S_J)$ as $n \to \infty$, where "$\Rightarrow$" is used to denote weak convergence. See [9, Section I.7], or note that $(nU^{(1)}, \cdots, nU^{(J)})$ has the same distribution as $(S_1, \cdots, S_J)(n/S_{n+1})$ [10, p. 335], so it suffices to note that by the law of large numbers, $n/S_{n+1} \Rightarrow 1$). This explains the appearance of S_j's in (6). In addition, the fact $nU^{(j)} \Rightarrow S_j$ implies that $U^{(j)}$ converges in probability to zero as $n \to \infty$ for j fixed, so that $a(U^{(j)}) \Rightarrow 1$ as $n \to \infty$ for j fixed.

Define, for $1 \leq k \leq J \leq n < \infty$, (for brevity, we suppress the variable k from the notation)

$$\begin{aligned} Z(J,n) = \left[\frac{nU^{(k)}}{a(U^{(k)})}\right]^{-1/\delta} - \frac{z}{1+z}\left\{\left[\frac{nU^{(1)}}{a(U^{(1)})}\right]^{-1/\delta} + \cdots \right. \\ \left. + \left[\frac{nU^{(J)}}{a(U^{(J)})}\right]^{-1/\delta} + n^{-1/\delta}N\right\} \end{aligned}$$

and, for $1 \leq J \leq \infty$

$$\tilde{Z}(J) = S_k^{-1/\delta} - \frac{z}{1+z}\{S_1^{-1/\delta} + \cdots + S_J^{-1/\delta}\}$$

so that $P(n,k,z,F) = P[Z(n,n) \geq 0]$ and $p(k,z,\delta) = P[\tilde{Z}(\infty) \geq 0]$. By the observations above, $Z(J,n) \Rightarrow \tilde{Z}(J)$ as $n \to \infty$, and by monotone convergence, $\tilde{Z}(J) \Rightarrow \tilde{Z}(\infty)$ as $J \to \infty$.

Suppose $\delta \geq 1$. Since $S_j/j \to 1$ with probability one, it follows that $\tilde{Z}(J) \to -\infty$ with probability one as $J \to \infty$. Given $\epsilon > 0$, fix J sufficiently large that $P[\tilde{Z}(J) \geq -1] \leq \epsilon$. Since $Z(J,n) \Rightarrow \tilde{Z}(J)$, it follows that if n is sufficiently large, then $P(n,k,z,F) \leq P[Z(J,n) \geq 0] \leq P[\tilde{Z}(J) \geq -1] + \epsilon \leq 2\epsilon$. Since $\epsilon > 0$ is arbitrary, $\lim_{n\to\infty} P(n,k,z,F) = 0$. Relation (4) is thus established if $\delta \geq 1$.

In the remainder of this section assume that $0 < \delta < 1$, for it remains to prove (4) in this case. The key is to bound the difference $R = Z(n,n) - Z(J,n)$

Lemma 7.2: Given $\epsilon, \eta > 0$, for any J sufficiently large, $P[R \geq \eta] \leq \epsilon$ *uniformly* in $n \geq J$.

Proof: Fix b with $0 < b < e^{-1}$ and integers n, J with $1 \leq J \leq n$, Observe that $nU^{(j)} \leq bj$ if and only if at least j of the uniform random variables are less than or equal to bj/n. Therefore, using $B(n,p)$ to denote a random variable with the binomial distribution of parameters n and p, $P[nU^{(j)} \leq bj] = P[B(n, bj/n) \geq j]$. Apply the two inequalities: $P[B(n,p) \geq j] \leq \binom{n}{j}p^j$ (this inequality follows by viewing $B(n,p)$ as the sum of n Bernoulli random variables, and noting that there are $\binom{n}{j}$ (overlapping) ways in which $B(n,p) \geq j$ can occur) and $\binom{n}{j} \leq (ne/j)^j$ to yield

$$P\big[nU^{(j)} \leq bj\big] \leq (ne/j)^j(bj/n)^j = (eb)^j. \tag{8}$$

Using the union bound technique, sum the right side of (8) over j with $J+1 \leq j \leq n$ to obtain

$$P\big[nU^{(j)} \leq bj \quad \text{for some } j, J+1 \leq j \leq n\big] \leq \frac{(eb)^{J+1}}{1-eb}. \tag{9}$$

Let $0 < \delta < 1$. Note that $R \geq 0$ with probability one. On the other hand, using (9), and setting $\bar{a} = \sup\{a(u) : 0 < u < 1\}$ yields

$$P\left[R \geq (\bar{a}/b)^{1/\delta} \sum_{j=J+1}^{\infty} j^{-1/\delta}\right] \leq \frac{(eb)^{J+1}}{1-eb}.$$

Bounding the sum on the left above by $\int_j^\infty x^{-1/\delta}dx$, and simplifying yield

$$P\left[R \geq \frac{\delta}{1-\delta}\left(\frac{\bar{a}}{bJ^{1-\delta}}\right)^{1/\delta}\right] \leq \frac{(eb)^{J+1}}{1-eb}. \tag{10}$$

Equation (10) implies the lemma. □

Note that $\tilde{Z}(\infty)$ has a continuous distribution function since $\tilde{Z}(\infty)$ is a strictly increasing function of the $k+1$st exponentially distributed random variable (which itself has a continuous distribution function) in the sequence defining $(S_j : j \geq 1)$. In addition, $\tilde{Z}(J) \Rightarrow \tilde{Z}(\infty)$, so that given $\epsilon > 0$ there exists $\eta > 0$ so that for all J sufficiently large

$$P[\tilde{Z}(J) \geq -2\eta] \leq P[\tilde{Z}(\infty) \geq 0] + \epsilon \tag{11}$$

$$P[\tilde{Z}(J) \geq 2\eta] \geq P[\tilde{Z}(\infty) \geq 0] - \epsilon. \tag{12}$$

By Lemma 7.2, taking J even larger if necessary guarantees that $P[R \geq \eta] \leq \epsilon$, or equivalently that

$$P[|Z(n,n) - Z(J,n)| \geq \eta] \leq \epsilon \tag{13}$$

for all $n \geq J$. Finally, take n so large that

$$P[|Z(J,n) - \tilde{Z}(J)| \geq \eta] \leq \epsilon. \tag{14}$$

Inequalities (13) and (14) and the triangle inequality imply that $P[|Z(n,n) - \tilde{Z}(J)| \geq 2\eta] \leq 2\epsilon$. Therefore, by (11)

$$\begin{aligned} P[Z(n,n) \geq 0] &\leq P[\tilde{Z}(J) \geq -2\eta] + 2\epsilon \\ &\leq P[\tilde{Z}(\infty) \geq 0] + 3\epsilon. \end{aligned}$$

and by (12)

$$\begin{aligned} P[Z(n,n) \geq 0] &\geq P[\tilde{Z}(J) \geq 2\eta] - 2\epsilon \\ &\geq P[\tilde{Z}(\infty) \geq 0] - 3\epsilon. \end{aligned}$$

Therefore, $|P[Z(n,n) \geq 0] - P[\tilde{Z}(\infty) \geq 0]| \leq 3\epsilon$. Since $\epsilon > 0$ is arbitrary, the proposition is proved.

Appendix C
Identification of the Limit

Expression (5) is verified in this Appendix, thereby completing the proof of Proposition 3.1. A by product of the proof of (4) given in Appendix B is that $p(k,z,\delta) = 0$ if $k \geq 1$ and $\delta \geq 1$. Consequently, $\phi(z,\delta) = 0$ for $\delta \geq 1$. Hence, for the remainder of this Appendix, suppose $0 < \delta < 1$.

Given $T \geq 0$, consider a new system that approximates the infinite one, defined as follows. Stations are distributed over the interval $[0,T]$ according to a Poisson process of unit intensity, and the received power for a station located at s is $s^{-1/\delta}$. Let $\phi_T(z,\delta)$ denote the corresponding mean number of captured transmissions. Since $\tilde{Z}(\infty)$ has a continuous distribution function, it is clear that $\phi(z,\delta) = \lim_{T\to\infty} \phi_T(z,\delta)$.

Well-known properties of the Poisson processes imply that $\phi_T(z,\delta)$ is equal to the mean number of stations, T, times the probability that a *test station* achieves capture. A test station is uniformly distributed over the interval $[0,T]$, and the interference power it encounters is W_T, the sum of the powers of all ordinary stations, given by $W_T = S_1^{-1/\delta} + \cdots + S_{N(T)}^{-1/\delta}$, where $N(T)$ is the number of ordinary stations. Thus, using a change of variable

$$\begin{aligned} \phi_T(z,\delta) &= T\left[\frac{1}{T}\int_0^T P\big[zW_T \leq s^{-1/\delta}\big]ds\right] \\ &= z^{-\delta}\delta\int_{T^{-1/\delta}/z}^{\infty} P[W_T \leq t]t^{-\delta-1}dt. \end{aligned} \tag{15}$$

We remark that the event $\{W_T \leq t\}$ occurs only if there is no ordinary arrival in the interval $[0, t^{-\delta}]$, so that $P[W_t \leq t] \leq \exp(-(T \wedge t^{-\delta}))$. From this, it follows that by taking ϵ sufficiently small, the contribution to (15) due to integration over $[T^{-1/\delta}/z, \epsilon]$ can be made arbitrarily small, uniformly in T for T sufficiently large. Thus, taking $T \to \infty$ in (15), and letting $W = \lim_{T\to\infty} W_T$ denote the sum of powers for the infinite system, yields that

$$\phi(z,\delta) = z^{-\delta}\delta\int_0^{\infty} P[W \leq t]t^{-\delta-1}dt. \tag{16}$$

Recall that the Gamma function is defined by $\Gamma(s) = \int_0^\infty x^{s-1}\exp(-x)dx$, so that

$$t^{-\delta-1} = \int_0^\infty \exp(-st)s^\delta ds/\Gamma(\delta+1). \tag{17}$$

Use (17) to substitute for $t^{-\delta-1}$ in (16), change the order of integration and integrate by parts to obtain

$$\begin{aligned} \phi(z,\delta) &= \frac{z^{-\delta}\delta}{\Gamma(\delta+1)}\int_0^\infty\int_0^\infty P[W \leq t]\exp(-st)dts^\delta ds \\ &= \frac{z^{-\delta}\delta}{\Gamma(\delta+1)}\int_0^\infty W^*(s)s^{\delta-1}ds \end{aligned} \tag{18}$$

where W^* is the Laplace transform of the density of W.

The random variable W_T is the sum of $N(T)$ random variables, each distributed as $U^{-1/\delta}$ for U uniformly distributed over $[0,T]$. Thus, the Laplace transform of the density of W_T is given by

$$\begin{aligned} W_T^*(s) &= \sum_{k=0}^{\infty}\frac{T^k\exp(-T)}{k!}\left\{\frac{\int_0^T \exp(-su^{-1/\delta})du}{T}\right\}^k \\ &= \exp\left[\int_0^T [\exp(-su^{-1/\delta}) - 1]du\right]. \end{aligned}$$

Letting again $T \to \infty$, and also using a change of variables, integration by parts, and the definition of the Gamma function, we obtain

$$\begin{aligned} W^*(s) &= \exp\left[\int_0^\infty [\exp(-su^{-1/\delta}) - 1]du\right] \\ &= \exp\left[\delta\int_0^\infty [\exp(-st) - 1]t^{-\delta-1}dt\right] \\ &= \exp\left[-s\int_0^\infty \exp(-st)t^{-\delta}dt\right] \\ &= \exp[-s^\delta\Gamma(1-\delta)]. \end{aligned}$$

Substituting this expression for W^* into (18), doing a final change of variables $v = s^\delta$, and using the fact $\Gamma(1+\delta)\Gamma(1-\delta) = \delta\Gamma(\delta)\Gamma(1-\delta) = \delta\pi\csc(\pi\delta)$ (see [11, eqs. (15) and (17)]) yields that

$$\begin{aligned} \phi(z,\delta) &= \frac{z^{-\delta}}{\Gamma(\delta+1)}\int_0^\infty \exp(-v\Gamma(1-\delta))dv \\ &= \frac{z^{-\delta}}{\Gamma(1+\delta)\Gamma(1-\delta)} \\ &= z^{-\delta}\mathrm{sinc}(\delta) \end{aligned}$$

which is the desired expression (5).

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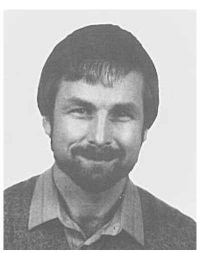

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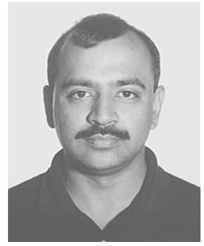

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