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ASYMPTOTIC ANALYSIS OF AN ASSIGMENT PROBLEM ARISING IN A DISTRIBUTED COMMUNICATIONS PROTOCOL

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Summary. Matchings for a random bipartite graph are considered. Each of the αM nodes on one side of the graph is directly connected to Q nodes chosen randomly and uniformly from among the M nodes on the other side of the graph. The size of matchings found by two simple approximation algorithms, as well as the size of the maximum matching when Q=2, are asymptotically determined in the limit as Q tends to infinity with α fixed. The work is motivated by a distributed communications protocol for accessing a silent receiver. The theory of approximating slow Markov random walks by ordinary differential equations is used for the analysis.

1. INTRODUCTION

A set of M *slots* and a set of *users* are given. Each user specifies a subset of slots, any one of which may eventually be assigned to the user. An *assignment* is a choice, for each slot, of at most one user from among the users that specify the slot, such that each user is chosen for at most one slot. The *size* of an assignment is defined to be the number of users that are assigned slots. The utilization of an assignment is the size of an assignment divided by M.

The setup just described corresponds to a bipartite graph as follows. The two sets of nodes of the graph are the set of users and the set of slots, and there is an edge between a particular user and a particular slot if and only if the slot was one of those specified by the user. An assignment corresponds to a matching, which is a set of node-disjoint edges, as illustrated in Fig. 1. The size of an assignment is the number of edges in the corresponding matching. The problem of finding a maximum size matching is called the bipartite matching problem[6].

Let Q be a positive integer. We will assume that the set of slots that any user can specify is chosen at random and is uniformly distributed over the $\binom{M}{Q}$ possibilities, and that the sets specified by different users are mutually independent.

This model arises in a simple packet radio network problem [7]. The users each wish to send a packet of information to a silent destination station. The time axis is divided into frames with M slots per frame. Each user transmits its packet Q times using spread-spectrum transmission, in Q randomly chosen slots. The receiver is assumed to know which slots each user will transmit in before the frame begins. The receiver computes an assignment, which is a choice of which user to listen to during each slot. Assuming perfect receiver selection capability, the size of the assignment is the number of successfully received packets. A related problem is discussed in [3].

We will consider three assignment strategies, described as follows:

Sequential-by-User Assignment Consider the users one at a time in an a priori fixed order. When a user is considered, first determine which of the slots it selected are not already assigned to other users. If there are any such slots, assign the user to one of them, with all such slots being equally likely. Then go on to the next user.

Sequential-by-Slot Assignment Suppose the slots are numbered and consider them in increasing order. When a slot is considered, first determine which as yet unassigned users have selected the slot. If there are any such users, assign the slot to one of them, with all such users being equally likely. Then go on to the next slot.

Maximum Assignment Compute a maximum size assignment using the well-known labeling algorithm for bipartite graphs [see 6, p. 221-222], which takes at most a constant times QM^2 computations.

The sequential-by-user strategy was used in [7], and a method was given there for computing the average utilization as a function of the number of users, the number of slots, and Q. Our goal in this paper is to consider the assignment strategies as the number of slots M tends to infinity, while the number of users is $[\alpha M]$ so that the number of users per slot tends to α. We denote the limit of the mean average utilization for the three assignment schemes by $U_u(\alpha,Q)$, $U_s(\alpha,Q)$, and $U_m(\alpha,Q)$ respectively. Propositions 1, 2 and 3, given in the next three sections, describe methods to readily compute these asymptotic average utilizations, though an expression for $U_m(\alpha,Q)$ is given only for Q=2. Numerical results are summarized in a table at the end of the paper. The method of analysis is the theory of "slow Markov chains"--in particular we appeal to a theorem in Kushner's book [5]. The earliest paper we know of that applies this method to a combinatorial problem is by Karp and Sipser [4]. The method is also used in [1] and [2].

2. ANALYSIS OF THE SEQUENTIAL-BY-USER ASSIGNMENT ALGORITHM

Let X(k) denote the number of users successfully assigned after k users have been considered, for $0 \le k \le M$. The average utilization is then $\dfrac{EX([M\alpha])}{M}$.

Proposition 1. Let $(x(t): 0 \le t \le \alpha)$ be determined by the ordinary differential equation (ODE) $x(0)=0$; $\dot{x}(t) = 1 - x(t)^Q$. Then

$$\lim_{M\to\infty} in\ prob. \max_{0\le k\le \alpha M} \left| \frac{X(k)}{M} - x(\frac{k}{M}) \right| = 0 \qquad (2.1)$$

and

$$U_u(\alpha,Q) = \lim_{M\to\infty} \frac{EX([M\alpha])}{M} = x(\alpha). \qquad (2.2)$$

Proof. Observe that $X(k+1) = X(k) + \xi_k$ where $\xi_k \in \{0,1\}$ and $\xi_k = 1$ if and only if not all Q slots specified bv user k+1 are among the X(k) slots already assigned. Clearly,

$$P[\xi_k=1 \mid X(k),\ldots,X(0)] = 1 - \frac{X(k)_Q}{M_Q} \qquad (2.3)$$

where we use the notation $r_Q = r(r-1)\ldots(r-Q+1)$ for $r>Q-1$ and $r_Q = 0$ otherwise. Hence X is a Markov chain with stationary transition probabilities[7]. We wish to apply the theory of slow Markov random walks, so we will change notation somewhat to match that of Kushner [5, Theorem 2, p. 108]. We define $\varepsilon = 1/M$ and $X_k^\varepsilon = \varepsilon X(k)$ and note that $X_{k+1}^\varepsilon = X_k^\varepsilon + \varepsilon\xi_k$ and for $0 \le x \le 1$,

$$P[\xi_k = 1 \mid X_k^\varepsilon = x] = 1 - \frac{(Mx)_Q}{M_Q}.$$

Since $r^Q \ge r_Q \ge (r-Q)_+^Q$ we see that

$$\lim_{\varepsilon\to 0} P[\xi_k = 1 \mid X_k^\varepsilon = x] = \lim_{\varepsilon\to 0} E[\xi_k \mid X_k^\varepsilon = x] = 1-x^Q.$$

All the conditions of [5, Theorem 2, p. 108] apply so that Eq. (2.1) is proved. Eq. (2.2) follows from Eq. (2.1) since $\frac{X([\alpha M])}{M}$ is bounded and converges to $x(\alpha)$ in probability.

3. ANALYSIS OF THE SEQUENTIAL-BY-SLOT ASSIGNMENT ALGORITHM

Let X(k) denote the number of slots successfully assigned after k slots have been considered. The average utilization of the sequential-by-slot algorithm is thus $\frac{EX(M)}{M}$.

Proposition 2. Let $(x(t), g(t): 0 \le t \le 1)$ be determined by the ODE

$$\dot{x} = 1 - e^{-g}$$
$$\dot{g} = -(1 - e^{-g})\frac{(Q-1)g}{Q(\alpha-x)} \qquad (3.1)$$

with $(x(0), g(0)) = (0, Q\alpha)$. Then

$$\lim_{M\to\infty} in\ prob \max_{0\le k\le M} \left|\frac{X(k)}{M} - x(\frac{k}{M})\right| = 0 \qquad (3.3)$$

and

$$U_s(\alpha,Q) = \lim_{M\to\infty} \frac{EX(M)}{M} = x(1).$$

Proof. Suppose the original model is changed so that each user specifies Q slots chosen independently and uniformly, with replacement. Hence, some users may select a slot more than once. However, the expected number of users that do so can easily be shown to be less than $\alpha Q^2/2$ for all M, so that for the purposes of proving Proposition 2 we can and do assume that the sampling with replacement model is used.

Suppose k slots have been considered so far. Let Y(k) denote the sum over the remaining M-k slots of the number of unassigned users that specify each slot. (If a user specifies a slot multiple times, the user is counted multiple times.) Let $\xi_1(k)$ denote the number of yet unassigned users that specify slot k+1. A user is assigned to slot k+1 if and only if $\xi_1(k) \ge 1$. If a user is assigned to slot k+1, consider the other Q-1 slots requested by the user, and let $\xi_2(k)$ denote the number of them that have not yet been considered. We then have, for $k \ge 0$

$$X(k+1) = X(k) + I_{\{\xi_1(k)\ge 1\}}$$
$$Y(k+1) = Y(k) - \xi_1(k) - \xi_2(k) I_{\{\xi_1(k)\ge 1\}}$$

Given (X(k), Y(k)), $\xi_1(k)$ has the binomial distribution with parameters $(Y(k), \frac{1}{n-k})$, and also given $(X(k), Y(k), \xi_1(k))$ with $\xi_1(k) > 0$, $\xi_2(k)$ has a distribution which is nearly binomial with parameters $(Q-1, \frac{Y(k)}{QM-X(k)})$. Thus, if we set $\varepsilon = \frac{1}{M}$, $X_k^\varepsilon = \varepsilon X(k)$, $Y_k^\varepsilon = \varepsilon Y(k)$, and $s_k^\varepsilon = \varepsilon k$ we have

$$\lim_{\varepsilon\to\infty} P[\xi_1(k) \ge 1 \mid X_k^\varepsilon = x, Y_k^\varepsilon = y, s_k^\varepsilon = s] = 1 - exp(-\frac{y}{1-s})$$

$$\lim_{\varepsilon\to\infty} E[-\xi_1(k) - \xi_2(k) I_{\{\xi_1(k)\ge 1\}} \mid X_k^\varepsilon = x, Y_k^\varepsilon = y, s_k^\varepsilon = s]$$

$$= -\frac{y}{1-s} - \frac{y}{Q(\alpha-x)}(Q-1)(1 - e^{-\frac{y}{1-s}})$$

We can apply [5, Theorem 2, p. 108] to yield that

$$\lim_{M\to\infty} in\ prob \max_{0\le k\le M} \left| X_k^\varepsilon - x(\frac{k}{M}) \right| + \left| Y_k^\varepsilon - y(\frac{k}{M}) \right| = 0$$

where

$$\dot{x}(s) = 1 - e^{-\frac{y}{1-s}}$$
$$\dot{y}(s) = -\frac{y}{1-s} - (1-e^{-\frac{y}{1-s}})\frac{y(Q-1)}{Q(\alpha-x)}$$

with $x(0) = 0$ and $y(0) = Q\alpha$. If we set $g(s) = \frac{y(s)}{1-s}$ we obtain the equivalent equations (3.1) for computing $(x(s): 0 \le s \le 1)$, and Eq. (3.2) is proved. Eq. (3.3) follows from Eq. (3.2) since $\frac{X(M)}{M}$ is bounded and converges to $x(1)$ in probability. □

4. ANALYSIS OF THE MAXIMUM ASSIGNMENT

The following proposition will be established in this section.

Proposition 3. Let Q=2 and let $\alpha > 0$. The asymptotic average utilization for an algorithm that produces maximum assignments is given by $U_m(\alpha, Q=2) = 1 - r + \alpha r^2$, where $r = \min\{r > 0 : r = exp(-2\alpha(1-r))\}$. In particular, $U_m(\alpha, Q=2) = \alpha$ for $0 \le \alpha \le \frac{1}{2}$.

A well-known algorithm [6, p. 222] efficiently finds maximum matchings in bipartite graphs. However, to only determine the maximum *size* of assignments an even simpler algorithm, which we describe next, can be used. The algorithm is defined as follows: While there exists a slot with exactly one unassigned user specifying the slot, chose one such slot at random and assign the corresponding user to the slot. It is not difficult to see that the assignments made by this algorithm are optimal. The algorithm may terminate before all the users are assigned, so the algorithm does not necessarily produce a maximum size assignment. However, we can still determine the maximum size of assignments using this algorithm in the case Q=2.

Indeed, let J(k) denote the set of unassigned slots that are requested by at least two users after k slots have already been assigned, and let k_o denote the number of slots assigned when the algorithm terminates. The following fact holds for the set $J(k_o)$ after the algorithm terminates: Given any subset A of $J(k_o)$, the sum over the slots in A of the number of requests for the slots by unassigned users is at least 2|A|, where |A| denotes the cardinality of |A|. Hence the number of unassigned users that request at least one slot in A is at least |A|. Since A is an arbitrary subset of $J(k_o)$, Hall's theorem for matching in bipartite graphs implies that more assignments can be made so that *all* slots in $J(k_o)$ are assigned. The size of a maximum matching is thus equal to $k_o + |A|$.

Turning now to the asymptotic analysis of the algorithm, define the following variables to represent the state after k slots have been assigned:

$X_1(k) =$ the number of unassigned slots with exactly one unassigned user specifying them

$X_2(k) =$ the sum, over all slots in $J(k)$, of the number of unassigned users that request the slot

$X_3(k) = |J(k)|$

and let $X(k) = (X_1(k), X_2(k), X_3(k))$.

If the algorithm has just completed assigning users to k slots, then the algorithm will continue if $X_1(k) > 0$. Suppose $X_1(k) > 0$ and consider the next user and slot to be assigned to each other. The assigned user also selected another alot, and we let $\xi(k)$ denote the number of other unassigned users in the other slot. Note that the other slot is in J(k) if and only if $\xi(k) \geq 1$. We have

$$X_1(k+1) = X_1(k) - 1 - I_{\{\xi(k)=0\}} + I_{\{\xi(k)=1\}}$$
$$X_2(k+1) = X_2(k) - I_{\{\xi(k)\geq 1\}} - I_{\{\xi(k)=1\}}$$
$$X_3(k+1) = X_3(k) - I_{\{\xi(k)=1\}}$$

Its not hard to see that

$$P[\xi(k) = 0 \mid X(k)] = \frac{X_1(k)}{X_1(k) + X_2(k)}$$

and that

$$P[\xi(k) = 1 \mid X(k)] = p(X_2(k), X_3(k)) P[\xi(k) \geq 1 \mid X(k)]$$

where p(a,b) is described as follows: If a distinct balls, $B_1, B_2, \ldots, B_a$, are placed into b distinct boxes in an independent, uniform fashion, and if all boxes are conditioned to have at least two balls in them, then p(a,b) is the probability that the box that contains ball B_1 contains exactly one other ball in it.

Lemma 4.1. Given $\delta > 0$, there exists M so large that $| p(a,b) - \overline{p}(\frac{a}{b}) | \leq \delta$ whenever $a \geq 2b \geq \delta M$ where

$$\overline{p}(\lambda) = \frac{\mu e^{-mu}}{(1-e^{-\mu})} \quad for\ \mu\ such\ that\ \lambda = \frac{\mu(1-e^{-\mu})}{(1-(1+\mu)e^{-\mu})}.$$

Proof of the lemma. The lengthy proof is omitted here. The proof is based on the fact that the conditional number of balls is a box is approximately Poisson distributed with mean μ, conditioned to be at least two. It follows that the number of balls in the box containing ball B_1 is approximately Poisson distributed with mean μ, conditioned to be at least one. □

Consider the ordinary differential equation

$$\dot{x}_1 = -1 + \frac{-x_1 + \overline{p}x_2}{x_1 + x_2}$$
$$\dot{x}_2 = \frac{-x_2(1+\overline{p})}{x_1 + x_2}$$
$$\dot{x}_3 = \frac{-\overline{p}\,x_2}{x_1 + x_2}$$

with $x(0) = (2\alpha e^{-2\alpha}, 2\alpha(1-e^{-2\alpha}, 1 - e^{-2\alpha}(1+2\alpha))$, where we write $\overline{p}$ for $\overline{p}(\frac{x_2}{x_3})$.

Let $\overline{t}$ be such that the solution to the above ODE is bounded away from the set $\{x_1 + x_2 = 0 \ or \ x_3 = 0\}$ over the time interval $[0, \overline{t}]$. Then [5, Theorem 2, p. 109] and Lemma 4.1 imply that

$$\lim_{m \to \infty} in\ prob. \max_{0 \leq k \leq M\overline{t}} \| \frac{X(k)}{M} - x(\frac{k}{M}) \| = 0.$$

Thus, the asymptotic average utilization is given by $U_m(\alpha, Q=2) = t_o + x_3(t_o)$ where $t_o = \min\{ t>0 : x_1(t) = 0\}$. It is straight forward to check that the above ODE has the following analytical solution, where $\mu(t) = 2\alpha(1 - \frac{t}{\alpha})^{\frac{1}{2}}$:

$$x_1(t) = \mu(\frac{\mu}{2\alpha} + e^{-\mu} - 1)$$
$$x_2(t) = \mu(1 - e^{-\mu})$$
$$x_3(t) = 1 - e^{-\mu}(1+\mu)$$

Note that $\mu(t)$ is monotone decreasing so that

$$\mu(t_o) = \max\{\mu > 0 : \frac{\mu}{2\alpha} + e^{-\mu} - 1 = 0\}, \tag{4.1}$$

and since $U_m(\alpha, Q=2) = t_o + x_3(t_o)$,

$$U_m(\alpha, Q=2) = \alpha - \mu(t_o) + \frac{\mu(t_o)^2}{4\alpha} + \frac{\mu(t_o)}{2\alpha}. \tag{4.2}$$

Upon reexpressing Eqs. (4.1) and (4.2) using the variable r defined by $\mu(t_o) = 2\alpha(1-r)$ we get Proposition 3. □

Remark. A generalization of the ODE above was given by Hajek and Cruz [1, p. 453] (though there was an error in their definition of $p_n(a,b)$--compare with p(a,b) here), who were interested in finding when $U_m(\alpha, Q) = \alpha$. The analytical solution that results in Proposition 3 is new, and similar expressions exist for the more general ODE considered in [1]. The observation that the simple algorithm determines the maximum size of assignments even if $U_m(\alpha, Q) \neq \alpha$ also appears to be new.

5. NUMERICAL RESULTS AND DISCUSSION

We computed the asymptotic utilization for the three assignment methods for several values of α and Q, using the expressions in Propositions 1-3. The results are displayed in Table 1. The utilizations for the sequential-by-user and the sequential-by-slot assignment methods do not significantly differ from each other. For Q=2 and α in the range $0.5 \leq \alpha \leq 1.0$, the maximum assignment is about 10% larger than that found by the one-pass algorithms, while for other values of α the difference is smaller. An obvious upper bound on the average utilization is $min(\alpha, 1)$, and it is interesting to note that utilizations for the sequential algorithms come fairly close to the bound for Q=3, except perhaps for α near one.

For each $Q \geq 2$ there is a value $\alpha_c(Q)$ so that $U_m(\alpha, Q) = \alpha$ if and only if $\alpha \leq \alpha_c(Q)$. Proposition 3 shows that $\alpha_c(2) = 0.5$. Cruz and Hajek [1, Prop. 2.1] used a moment method to provide a lower bound on $\alpha_c(Q)$ for all Q. The bound tends to one as Q tends to infinity. For Q=3,4,5,6 or 7, the lower bounds are 0.65, 0.86, 0.94, 0.97 and 0.99, respectively. Thus, for example, if Q=4 and the number of users per slot is less than 0.86, then as M tends to infinity the proportion of users that do not get slot assignments tends to zero if an optimal assignment algorithm is used. Propositions 1 and 2 show that there is no such positive critical value of α for the two one-pass sequential strategies.

Finally, we will return to the discussion started in the introduction about the application to distributed reservation-based access to a silent receiver. In addition to computing the average utilization for the sequential-by-user strategy, Wieselthier et. al [7] also computed the effects of multiple simultaneous transmissions under a more realistic model in which the spread-spectrum signaling does not provide a perfect capture capability. The key to their analysis is to compute the distribution of the number of other transmissions in progress (called secondary interference in [3]) while the receiver is attempting to receive a typical packet. The degradation caused by the other transmissions is the main reason for the users to use a small value of Q. One might consider assignment strategies in which the amount of other interference in all slots is taken into account in the assignment process. (This is done for a related problem in [3].) Though the analysis may tend to be more difficult, we can expect that the performance of very simple strategies will not be far from optimal, if the analysis in this paper is any indication.

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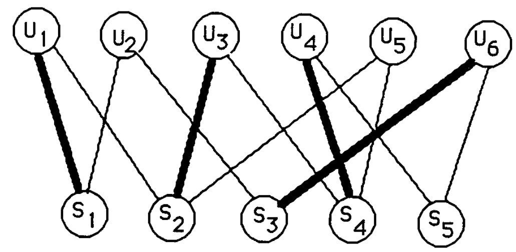

Figure 1. An example of an assignment. Six users each selected two out of the five slots. The assignment consists of the set of boldface edges. This corresponds to M=5, Q=2, α=1.2, and utilization equal to 0.8. The assignment does not have maximum size.

Table 1. The asymptotic utilization achieved by the maximum assignment (for Q=2 only), the sequential-by-user assignment, and the sequential-by-slot assignment algorithms, as computed using Propositions 1-3. Each user selects Q slots at random, and the number of users per slot is α.

Q=2

α	U_m	U_u	U_s
0.10	0.10000	0.0997	0.0997
0.20	0.20000	0.1974	0.1979
0.30	0.30000	0.2913	0.2927
0.40	0.40000	0.3800	0.3826
0.50	0.50000	0.4621	0.4662
0.60	0.59630	0.5371	0.5425
0.70	0.67839	0.6044	0.6110
0.80	0.74452	0.6641	0.6716
0.90	0.79686	0.7163	0.7244
1.00	0.83810	0.7616	0.7699
1.50	0.94579	0.9052	0.9111
2.00	0.98096	0.9640	0.9670
2.50	0.99314	0.9866	0.9879
3.00	0.99750	0.9951	0.9956
3.50	0.99909	0.9982	0.9984
4.00	0.99966	0.9993	0.9994

Q=3

α	U_u	U_s
0.10	0.1000	0.1000
0.20	0.1996	0.1998
0.30	0.2980	0.2987
0.40	0.3938	0.3956
0.50	0.4852	0.4889
0.60	0.5703	0.5763
0.70	0.6476	0.6559
0.80	0.7157	0.7259
0.90	0.7742	0.7855
1.00	0.8231	0.8346
1.50	0.9545	0.9605
2.00	0.9895	0.9913
2.50	0.9976	0.9981
3.00	0.9995	0.9996
3.50	0.9999	0.9999
4.00	1.0000	1.0000

Q=4

α	U_u	U_s
0.10	0.1000	0.1000
0.20	0.1999	0.2000
0.30	0.2995	0.2998
0.40	0.3980	0.3989
0.50	0.4939	0.4962
0.60	0.5853	0.5899
0.70	0.6696	0.6770
0.80	0.7443	0.7544
0.90	0.8078	0.8195
1.00	0.8592	0.8712
1.50	0.9770	0.9814
2.00	0.9968	0.9976
2.50	0.9996	0.9997
3.00	0.9999	1.0000
3.50	1.0000	1.0000
4.00	1.0000	1.0000

Q=5

α	U_u	U_s
0.10	0.1000	0.1000
0.20	0.2000	0.2000
0.30	0.2999	0.3000
0.40	0.3993	0.3997
0.50	0.4974	0.4987
0.60	0.5925	0.5955
0.70	0.6818	0.6875
0.80	0.7619	0.7706
0.90	0.8295	0.8405
1.00	0.8830	0.8945
1.50	0.9880	0.9908
2.00	0.9990	0.9993
2.50	0.9999	0.9999
3.00	1.0000	1.0000
3.50	1.0000	1.0000
4.00	1.0000	1.0000