

Stochastic Approximation Methods for Decentralized Control of Multiaccess Communications

Invited Paper

BRUCE HAJEK, MEMBER, IEEE

Abstract—Strategies based on stochastic approximation are suggested for controlling the transmission probabilities of stations sharing a collision access communication channel. Examples include strategies based on acknowledgment feedback only, and strategies suitable for frequency-hopped spread spectrum channels. The strategies provide for the incorporation of priorities among the stations. A Lyapunov-like function is constructed to prove global stability of the ordinary differential equation which is approximated by the system, and a design methodology based on minimizing a cost function is given. Preliminary results on transmission control in a nonbroadcast network are presented.

I. INTRODUCTION

THE CLASSICAL ALOHA random access strategy is a viable one in many situations, but stable operation requires that transmission probabilities be updated [5], [18]. The first approach suggested for stabilizing ALOHA was to let transmission probabilities be a function of the number of backlogged stations [5], [6], [14]. However, the stations may not know the number of backlogged stations, so the next logical step was for the stations to base their transmission probabilities on *estimates* of the number of backlogged stations. This approach was first taken by Lam and Kleinrock [14] in their control algorithm called CONTEST. Their simulation results for finite-user models were impressive, even though they used a primitive estimation procedure. An and Gelenbe [1] and Seret and Macchi [23] presented similar results for an infinite-population model. In [1], a finite-dimensional approximation to the exact recursive backlog estimators derived by Segall [21] was used, while Seret and Macchi [23] estimated the channel traffic level itself instead of estimating how many stations were generating the traffic. These control strategies are based on 0-1- e feedback information (defined in Section III, Example 1).

Mikhailov [16], [17] and, later and independently, Hajek and van Loon [8] gave transmission control strategies based on 0-1- e feedback which they proved achieved stable throughput at rates up to e^{-1} packets/slot. Cruz [4]

obtained a proof of stability of the dynamic retransmission control strategies with *noisy* 0-1- e feedback by constructing a Lyapunov function on a two-dimensional statespace (backlog \times transmission probabilities). Cain [3] reported extensive simulations using dynamic transmission strategies for systems that are subjected to severe variations in traffic load.

Dynamic strategies for controlling transmission probabilities using only acknowledgment feedback (defined in Section III, Example 2) were studied in [9], [12], and [25]. Although it is still an open problem to find a random access strategy which uses acknowledgment information only and which maintains stability for the Poisson infinite-population user model, some progress has been made. Kumar and Merakos [25] have shown that acknowledgment feedback can be used to eliminate undesirable bistable behavior under finite user one-packet-at-a-time models. Kelley [12] presented a clever argument showing, roughly speaking, that the time between successive attempts by a blocked packet must grow exponentially if there is to be any chance for stability in the infinite population case. Since acknowledgment feedback is less than 0-1- e feedback, it is known that stable throughput cannot exceed 0.587 packets/slot [26].

In the work described above [8], [16], [17], $(N(k), f(k))$ forms a Markov chain, where $N(k)$ is the number of backlogged stations and $f(k)$ is the retransmission probability used in slot k . In [8], this chain is called "the global model." To study the global model, the dynamics of the retransmitted traffic intensity, $\mu(k) = N(k)f(k)$, are considered when the backlog $N(k)$ is artificially fixed at some level n . It turns out that if the multiplicative update rule,

$$f(k+1) = f(k) \times (\text{function of channel feedback in slot } k),$$

is used, then the dynamics of the product $N(k)f(k)$ do not depend strongly on n ; there is a single chain, called the "local model" in [8], which models $\mu(k)$ when the backlog is fixed at any given level. The goal of the retransmission strategy is to steer $\mu(k)$ towards a desirable value μ^* . $N(k)$ in fact varies with k in the global model, and an exact analysis of the two-dimensional chain $(N(k), f(k))$ appears prohibitive. One technique for approximate analysis

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The author is with the Department of Electrical and Computer Engineering and the Coordinate Science Laboratory, University of Illinois at Urbana-Champaign, 1101 West Springfield Avenue, Urbana, IL 61801, USA.

is to view the variations of $N(k)$ as inducing a disturbance on the transmitted traffic process $\mu(k)$. The effect of such a disturbance can be predicted by finding, in the *undisturbed* local model,

- 1) the rate of convergence of $\mu(k)$ to a neighborhood of μ^* given a large initial separation, and
- 2) the steady state accuracy of $\mu(k)$ as an approximation to μ^* .

These two important quantities can be determined in a precise asymptotic sense using the theory of diffusion approximations, a well refined tool in the theory of stochastic approximation schemes [13]. This procedure was initiated in [8]. Subsequently, Merakos and Kazakos [15] made an extensive study of the local model under a variety of modifications of the channel protocol and showed how certain parameters should be chosen. The really crucial and difficult problem in this program of analysis, which was only touched on in [8] and was not addressed in [15] nor will it be here, is how to compute delay accurately in the global model.

In much of the work on random access protocols, it is assumed that stations can accumulate at most one packet at a time. In some situations, such as on a reservation channel, this assumption might be valid. In other situations, however, one would like to consider random access systems in which each station can queue many packets. While promising iterative approximation schemes have been developed to analyze such complex models [20], [22], [24], it is difficult to justify all the assumptions used (e.g., independent arrival processes at distinct stations) and to check the accuracy of the approximations. That, finally, is where this paper fits in. We focus on the control of transmission probabilities when there are N stations that each have a need for steady throughput, and N is constant. Thus, the model we analyze can be thought of as a local model that can perhaps be used to study "global" models in which N is varying at moderate rates because of stations becoming empty and new stations receiving packets. Using standard techniques from stochastic approximation theory, we can determine roughly how fast the number of stations can be allowed to fluctuate in order for our estimators to track the backlog (this is determined by the speed of convergence of the estimators for fixed n) and how accurate the estimators will be for such varying N . This is illustrated in [2].

For the case of a fixed number N of busy stations (which, strictly speaking, is all that we analyze in this paper), it is clear that a superior protocol is time-division multiple-access (TDMA) or, for example, Itai and Rosberg's clever variation of TDMA [11] which accommodates stations with unequal demands. By "continuity," it is clear that, even if N is not constant but is only very slowly changing, then an adaptive variant of TDMA would still be superior to the schemes that we analyze. The schemes that we suggest, however, can be used even in cases where N changes relatively quickly, and our analysis for fixed N is relevant to the case of time-varying N . We suggest, for

example, a scheme based on acknowledgment feedback only; it is not clear how adaptive TDMA can be made with such limited feedback.

As hinted in Section III, Example 3, the stochastic approximation methodology used in this paper can readily be applied to the control of transmission probabilities in spread-spectrum networks. Other problems, such as control of code rates, transmission range, and bandwidth, can also be fruitfully addressed using stochastic approximation.

We begin Section II by introducing a general class of control strategies and proving a stability result for them. Examples, including some numerical analysis, are presented in Section III. A method for assigning priorities is presented in Section IV, and in Section V we briefly consider a situation in which communication links are interfered with by only some of the other links.

II. CONTROL IN A BROADCAST ENVIRONMENT

Consider N stations sharing a communication channel. During time slot k , station j transmits a packet with probability $f_j(k)$. We set $U_j(k) = 1$ if j transmits in slot k and $U_j(k) = 0$ otherwise. At the end of the slot, station j receives channel feedback, which is represented by a random variable $Y_j(k)$. We assume that, given $f(k)$, the random vector $(U(k), Y(k))$ is conditionally independent of the past, and that the conditional distribution of $(U(k), Y(k))$ given $f(k)$ is not dependent on k . We will consider certain transmission control strategies of the form

$$f_j(k+1) = \left[f_j(k) + \epsilon G_j(f_j(k), Y_j(k), U_j(k)) \right]_0^{\Delta_j}, \quad (2.1)$$

where $\Delta = (\Delta_1, \dots, \Delta_N)$ is a vector with positive components, $[x]_a^b$ denotes the projection, $\min\{b, \max\{a, x\}\}$, of x onto the interval $[a, b]$, and the G_j 's are bounded continuous functions.

By our assumptions, there is an N -vector valued function \bar{G} defined by

$$\bar{G}_j(f) = E \left[G_j(f_j(k), Y_j(k), U_j(k)) \mid f(k) = f \right]. \quad (2.2)$$

Define a continuous-time random process f^ϵ by setting $f^\epsilon(t) = f(k)$, whenever $k\epsilon = t$ for an integer k , and by requiring the sample paths of f^ϵ to be affine over intervals of the form $[k\epsilon, (k+1)\epsilon]$. Then, by the well-known theory of stochastic approximation, under quite general conditions (essentially, all that is needed is that the conditional distribution of $(U(k), Y(k))$ given $f(k)$ should depend smoothly on f [13, Sects. 5.3, 5.4 and 4.4]) the random process f^ϵ converges weakly as ϵ tends to zero to the solution of the following ordinary differential equation (ODE) with boundary condition

$$\dot{f} = \bar{G}(f) + v + u, \quad f(0)$$

given

$$\begin{aligned} 0 \leq f(t) \leq \Delta, \quad v(t) \leq 0, \quad u(t) \geq 0, \\ v_i(t)(\Delta_i - f_i(t)) = 0, \quad u_i(t)f_i(t) = 0 \end{aligned} \quad (2.3)$$

where u and v are auxiliary functions to be determined together with f . This convergence result is akin to the law of large numbers. A second convergence result, akin to the central limit theorem, pertains to the limiting distribution as ϵ tends to zero of the random process \tilde{f}^ϵ defined by

$$\tilde{f}^\epsilon(t) = \frac{f^\epsilon(t) - f^*}{\sqrt{\epsilon}},$$

where f^* is a locally and asymptotically stable equilibrium point of the ODE (2.3). For simplicity, we assume that $0 < f_j^* < \Delta_j$ for each j . Then, under rather general conditions [13, Sects. 5.6 and 4.9], if $\tilde{f}^\epsilon(0)$ converges weakly (as ϵ tends to zero) to a constant $x(0)$, or if f^* is globally and asymptotically stable and an initial transient period is ignored, \tilde{f}^ϵ converges weakly to a Gaussian diffusion process described by the Itô stochastic differential equation

$$dx(t) = Fx(t) + \Sigma^{1/2}dw.$$

Here w is a standard N -dimensional Wiener process,

$$F = \left. \frac{\partial \bar{G}}{\partial f} \right|_{f^*} \text{ and } \Sigma = \text{cov}(\xi | f(k) = f^*), \quad (2.4)$$

where

$$\xi_j = G_j(f_j(k), Y_j(k), U_j(k)).$$

The invariant measure for $(x(t))$ is Gaussian with mean zero and covariance matrix Q satisfying

$$QF + FQ + \Sigma = O. \quad (2.5)$$

If the diffusion approximation is sufficiently accurate, the covariance matrix of $f(k)$ for large k should thus be $Q\epsilon + o(\epsilon)$.

The eigenvalues of F give information on the asymptotic convergence rate of solutions to the ODE. The more negative the real parts of the eigenvalues, the faster the convergence rate. In fact, ϵ times the negative real part of an eigenvalue serves as an inverse time constant for the part of f in the direction of the corresponding eigenvector. The matrix Q gives a measure of steady state accuracy of the control strategy. In particular, $(a^T Q a)\epsilon$ gives the approximate steady state variance of $a \cdot f$ for any constant N vector a .

We will consider strategies such that

$$\bar{G}_j(f) = (f_j)^r (\phi_N(f) - \alpha A_j(f_j)), \quad O \leq f \leq \Delta \quad (2.6)$$

where the exponent r is a constant with $r \geq 1$, α is a strictly positive constant, and the following conditions are satisfied for all j .

- C.1) $\phi_N(f)$ is strictly decreasing in f and is continuously differentiable.
- C.2.j) $A_j(0) = 0$, $A_j(u)$ is strictly increasing in u and is continuously differentiable, and $u'A_j(u)$ has a finite right hand derivative at $u = 0$.

Examples are considered in the next section.

These conditions ensure that \bar{G} is locally Lipschitz continuous so that the ODE (2.3) has a unique solution for any

initial point. We will consider the ODE (2.3) restricted to the region

$$\Omega = \{f: 0 < f_j \leq \Delta_j \text{ for each } j\}.$$

Since $\dot{f}_j \geq cf_j + o(f_j)$ for all j and for some possibly negative constant c , the solution $f(t)$ remains in Ω for all finite t . Thus, the auxiliary variables $u_i(t)$ in (2.3) are identically zero, so the ODE (2.3) is equivalent to

$$\begin{aligned} \dot{f}_j &= (f_j)^r (\phi_N(f) - \alpha A_j(f_j)) + v_j, \\ v(t) &\leq 0, v_i(t)(\Delta_i - f_i(t)) = 0. \end{aligned} \quad (2.3')$$

Many stronger convergence and stability properties can be proved under a wide variety of general assumptions for the estimators that we consider. Generally, such results are straightforward to prove, although the proofs are long and technical, if a Lyapunov (or similar) function can be explicitly given which can be used to establish global asymptotic stability for the ODE [13, Ch. 6] and references therein). On the other hand, it is very difficult to find a suitable global Lyapunov function for many nonlinear control problems. Because of these facts, we limit ourselves in this paper to presenting a Lyapunov-style proof of global stability for the ODE (2.3).

Proposition 1: Consider the ODE (2.3) restricted to the region Ω . Suppose that Condition C.1 and Condition C.2.j are satisfied for each j , and suppose that $\phi_N(O) > 0$. Then there exists a unique point f^* in Ω such that, starting from any point in Ω , the trajectory $(f(t))$ converges to f^* as t tends to infinity (i.e., there is a unique globally and asymptotically stable point f^* in Ω).

Lemma 1: There is a unique equilibrium point f^* in Ω .

Proof: The Kuhn-Tucker condition, which characterizes an equilibrium point f in Ω , demands that

$$\phi_N(f) - \alpha A_j(f_j) \geq 0,$$

for all j , with equality for j such that $f_j < \Delta_j$. An equivalent condition is that $f_j = r_j(\phi_N(f))$ for each j , where the r_j are nondecreasing functions defined by

$$r_j(\theta) = \begin{cases} \Delta_j, & \text{if } \theta \geq \alpha A_j(\Delta_j) \\ A_j^{-1}(\theta/\alpha), & \text{if } 0 \leq \theta \leq \alpha A_j(\Delta_j). \end{cases}$$

Thus, given $\theta \geq 0$, there is an equilibrium point f in Ω with $\phi_N(f) = \theta$ if and only if

$$\theta = \phi_N(r_1(\theta), \dots, r_N(\theta)).$$

The right side of this equation is nonincreasing in θ and is strictly positive for $\theta = 0$. There is thus a unique strictly positive solution, and hence there is a unique equilibrium point f^* in Ω .

Lemma 2: Define subsets A and B of Ω by

$$\begin{aligned} A &= \{f: \phi_N(f) - \alpha A_j(f_j) \leq 0 \text{ for some } j\}, \\ B &= \{f: \phi_N(f) - \alpha A_j(f_j) \geq 0 \text{ for some } j\}. \end{aligned}$$

Then A and B are invariant for the ODE.

Proof: By a standard perturbation trick [19, p. 348] based on the uniqueness of solutions to the ODE and on the fact that A and B are closed in Ω , we can and do assume without loss of essential generality that the derivatives of the functions A_j are *strictly* positive and that the first partial derivatives of the function ϕ_N are *strictly* negative. For the sake of argument by contradiction, we suppose that some trajectory ($f(t)$) starts in A but is not always contained in A . Since A is closed in Ω , there exists a time t^* such that $f(t^*)$ is in the boundary of A and $f(t^* + \delta)$ is not in A for a sequence of positive δ 's converging to zero. Set $u = f(t^*)$. Since u is on the boundary of A ,

$$\phi_N(u) - \alpha A_j(u_j) \geq 0 \quad \text{for all } j \quad (2.7)$$

and equality holds for *some* value j_0 of j . On the other hand, equality cannot hold for all values of j such that $u_j < \Delta_j$, for otherwise u would be an equilibrium point so that $f(t) = u$ for all t larger than t^* , a contradiction. Therefore, the inequality in (2.7) is strict for at least one value of j such that $u_j < \Delta_j$. Thus, $\dot{f}_j(t^*)$ is nonnegative for all j (by (2.7)) and is strictly positive for at least one value of j . This implies in turn that $\dot{\phi}(f(t^*))$ is strictly negative so that the right-hand derivative of $\phi_N(f(t)) - \alpha A_{j_0}(f_{j_0}(t))$ at t^* is strictly negative. Thus, $\phi_N(f(t^* + \delta)) - A_{j_0}(f_{j_0}(t^* + \delta))$ is strictly negative, and hence $f(t^* + \delta) \in A$ for all sufficiently small, positive δ . This contradicts our choice of t^* , so that A must be an invariant set. The proof that B is invariant is similar.

Define functions V^+ , V^- and V on Ω by

$$\begin{aligned} V^+(u) &= \max_j A_j(u_j), \\ V^-(u) &= \min_i A_i(u_i), \\ V(u) &= V^+(u) - V^-(u) \end{aligned}$$

and, for a given trajectory, define

$$\begin{aligned} V_t^+ &= V^+(f(t)) \\ V_t^- &= V^-(f(t)) \\ V_t &= V(f(t)). \end{aligned}$$

Lemma 3: V_t^+ is nonincreasing (resp. V_t^- is nondecreasing) after, if ever, the trajectory enters A (resp. B). V_t is nonincreasing after, if ever, the trajectory enters $A \cap B$. V_t is strictly decreasing on the time set

$$\{t: f(t) \in A \cap B \text{ and } V_t > 0\}. \quad (2.8)$$

Proof: Suppose that $f(t) \in A$ for $t \geq t^*$. For each $t \geq t^*$, let

$$J(t) = \{i: A_i(f_i(t)) = V_t^+\}.$$

Then $\dot{f}_i(t) \leq 0$ for all i in $J(t)$ and $t \geq t^*$. Thus, using D^+ to denote upper right-hand Dini derivatives with respect to time, we have

$$D^+ V_t^+ = \max \{D^+ A_i(f_i(t)): i \in J(t)\} \leq 0,$$

which implies that V_t^+ is nonincreasing for $t \geq t^*$ [19, p. 348]. A similar argument shows that (V_t^-) is nondecreasing

after the trajectory enters the set B , and it then follows that (V_t) is nonincreasing after the trajectory enters $A \cap B$.

To prove the final statement, note that the set in (2.8) is contained in the union of $\{t^*\}$ and the two open sets

$$\begin{aligned} \{t: t > t^*, \alpha V_t^+ > \phi_N(f_t)\}, \\ \{t: t > t^*, \alpha V_t^- < \phi_N(f_t)\}, \end{aligned}$$

where t^* is the first time that the trajectory is in $A \cap B$. (V_t^+) is strictly decreasing on the first, and (V_t^-) is strictly increasing on the second of these two open sets, and this implies the lemma.

Proof of Proposition 1: Consider a solution ($f(t)$) to the ODE (2.3) for some initial point in Ω . Since f cannot return to A^c or B^c once it has left, at least one of the following three cases must hold.

- (i) f remains in A^c for all time,
- (ii) f remains in B^c for all time,
- (iii) f enters $A \cap B$ in finite time.

In the first case, each f_j is strictly increasing, and hence converges to a strictly positive point that, by continuity, must be an equilibrium point in Ω . The trajectory thus converges to f^* .

In the second case, each f_j is strictly decreasing and thus converges. Since the zero vector is not a limit point of B^c , at least one f_j must have a strictly positive limit. This implies that $\phi_N(f)$ also has a strictly positive limit and thus, finally, that all the f_j have a strictly positive limit. Thus, as in the first case, f converges to the unique equilibrium point in Ω , f^* .

In the third case, V_t^- is ultimately nondecreasing and V_t is ultimately nonincreasing. The limit set of the trajectory is thus contained in the compact set

$$A \cap B \cap \{u: V^-(u) \geq \delta\}$$

for some positive δ , and (V_t) converges to some nonnegative constant c . By the continuity of V , the limit set is also contained in the subset of Ω where $V(f) = c$. Since the limit set is invariant [19, p. 367] and since V is strictly decreasing along any trajectory in the set $A \cap B \cap \{V(f) > 0\}$, it must be that $c = 0$. Thus, the limit set is contained in the set $A \cap B \cap \{V(f) = 0\}$, which contains only the single point f^* . The set is thus equal to $\{f^*\}$, and the proposition is proved.

III. EXAMPLES

Three examples of transmission strategies of the form (2.1) are given in this section such that \bar{G} has the form in (2.6) where ϕ_N and the functions A_j satisfy Condition C.1) and the Conditions C.2.j). Roughly speaking, since the term ϕ_N in (2.6) does not depend on j , it serves to coordinate the stations. The key to finding strategies such that (2.6) is satisfied is identifying ways to introduce the common term ϕ_N . The designer typically has complete flexibility in choosing the functions A_j , and a guideline for choosing them is discussed in the next section.

Example 1—Common 0-1-e Feedback: Suppose that

$Z(k) = 0$ if no transmissions,

$Z(k) = 1$ if one transmission, and

$Z(k) = e$ if more than one transmission (3.1)

occur in slot k , and suppose that $Y_j(k) = Z(k)$ for all j . Thus, each station learns whether zero, one or more than one transmission occurs in the slot. Consider a transmission rule of the form in (2.1) where

$$G_j(f, y, U) = f_j [c(y) - \alpha A_j(f_j)]$$

for some vector $c = (c(0), c(1), c(e))$ and for arbitrary functions A_j satisfying Condition C.2.j) for all j . Then \bar{G} has the form in (2.6) with

$$\phi_N(f) = c \cdot (\pi_0(f), \pi_1(f), \pi_e(f))$$

where

$$\pi_0(f) = \prod_i (1 - f_i), \quad \pi_1(f) = \sum_i f_i \prod_{j \neq i} (1 - f_j)$$

and $\pi_e(f) = 1 - \pi_0(f) - \pi_1(f)$. Condition C.1) is satisfied as long as $c(0) \geq c(1) \geq c(e)$ with at least one of the inequalities being strict (unless $N = 1$, in which case we need $c(1) > c(0)$).

In order to choose a vector c , we consider the Poisson approximation: if the number of packets transmitted in a slot were Poisson-distributed with mean G and if Z were defined as in (3.1), then $c(Z)$ would have expected value

$$c \cdot (e^{-G}, Ge^{-G}, 1 - (1 + G)e^{-G}). \quad (3.2)$$

We now decide that it is desirable to maximize the probability of success and therefore to have $G = 1$. We thus choose c so that the quantity in (3.2) is positive (resp. negative) if $G < 1$ (resp. $G > 1$). These conditions dictate [8] that $c \neq 0$, $c(e) \leq 0 \leq c(0)$, and that the quantity in (3.2) is zero for $G = 1$. Numerical results for this strategy when $c = (0.418, 0, -0.582)$ are given after the next example.

Example 2—Acknowledgment-Based Transmission Control: Essentially the only information required by a station to implement the strategy we will give next is whether or not its own transmissions are successful. A station receives no information from slots where it is inactive. The idea behind this strategy begins with the observation that the probability that no station transmits in slot k is given by

$$p(k) = \prod_i (1 - f_i(k)). \quad (3.3)$$

If station j decides to transmit in slot k , it can form an unbiased estimate of $p(k)$ since

$$p(k) = p_j(k)(1 - f_j(k)) \quad (3.4)$$

where $p_j(k)$ is the conditional probability that station j is successful, given that it transmits in slot k .

Roughly speaking, the stations can thus control $p(k)$. We again turn to the case of Poisson traffic in order to determine how the stations might use this ability. If the number of transmissions in the slot had a Poisson distribution with mean G , then the success probability Ge^{-G} is

maximized when $G = 1$, or equivalently, when the probability of no transmissions is equal to $1/e$. This suggests that we declare that the goal of the transmission control strategy is to drive $p(k)e$ close to one. We are led to consider the following update rule.

$$f_j(k+1) = [f_j(k) + \epsilon f_j(k) c_j(f_j(k), Z(k), U_j(k))] \Delta_0$$

where $\Delta = 1 - e^{-1}$,

$$c_j(u, Z, U) = \begin{cases} (1-u)e - 1 - \alpha A_j(u), & Z = 1, U = 1 \\ -1 - \alpha A_j(u), & Z = e, U = 1 \\ 0, & U = 0 \end{cases}$$

and the functions A_j are arbitrary functions satisfying Conditions C.2.j) for each j .

We find that

$$\begin{aligned} \bar{G}_j(f) &= E [f_j(k) c_j(f_j(k), Z(k), U_j(k)) | f(k) = f] \\ &= f_j(k)^2 E [c_j(f_j(k), Z(k), U_j(k)) | f(k) = f, U_j(k) = 1] \\ &= f_j(k)^2 \{ p_j(k)(1 - f_j(k))e - 1 - \alpha A_j(f_j(k)) \} \end{aligned}$$

and thus we see, using (3.3) and (3.4), that \bar{G} has the form of (2.6) with $r = 2$ and

$$\phi_N(f) = \left\{ e \prod_j (1 - f_j) \right\} - 1.$$

A. Numerical Study of Examples 1 and 2

For simplicity, we assume that the functions A_i are the same for all i . Then the globally stable equilibrium point f^* of the ODE (2.3) has the form $f^* = (u, \dots, u)$, where u is easily computed. The matrices F and Σ defined in (2.4) have the form

$$F = aI + bW \quad \Sigma = cI + dW$$

where a , b , c , and d are readily computed and where W is the matrix of all ones. Thus the scaled covariance matrix Q determined by (2.5) also has the form $Q = rI + sW$, where r and s can be readily expressed in terms of a , b , c , and d . In general, a and b are nonpositive. F has eigenvalue $a + b$ with multiplicity one (corresponding to the eigenvector of all ones), and eigenvalue a with multiplicity $n - 1$.

Until otherwise stated, we will assume that $A_i(u) = u$ for each i . In Fig. 1, we picture the throughput evaluated when the transmission probabilities are equal to their equilibrium value f^* . For N stations using equal transmission probabilities, the throughput achieves its maximum value of $(1 - 1/n)^{n-1}$ when $f = 1/n$. For both Examples 1 and 2, the equilibrium transmission probabilities are slightly smaller than $1/n$ and are decreasing in α . However, we see from Fig. 1 that, for α smaller than about 0.5, the throughput at the equilibrium points is close to the maximum possible (for each n). If, for example, $\alpha < 0.5$, then the throughput remains above e^{-1} for all n .

In Fig. 2 we plot the value of the diagonal entries of Q and in Fig. 3 we plot the smallest magnitude eigenvalue of F , all as functions of α . To illustrate the use of these

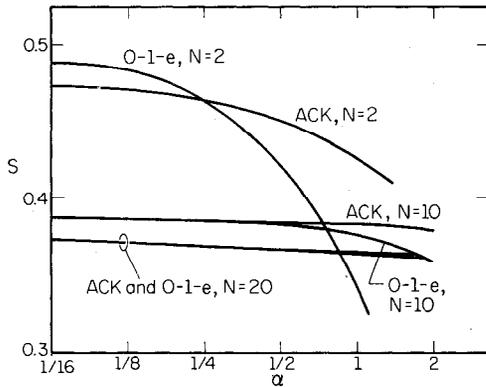


Fig. 1. Throughput summed over all stations evaluated at equilibrium vector f^* , for Examples 1 and 2.

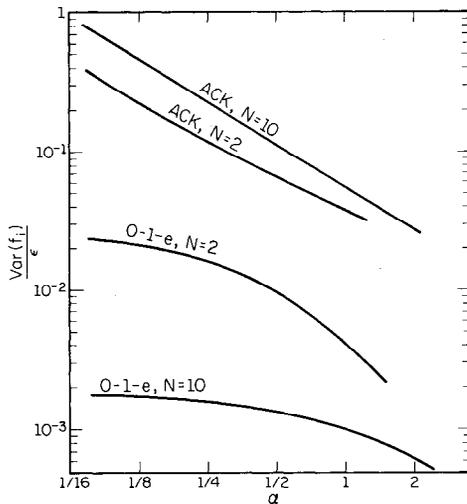


Fig. 2. Scaled asymptotic variance (for ϵ tending to zero) of individual transmission probabilities f_i , in equilibrium.

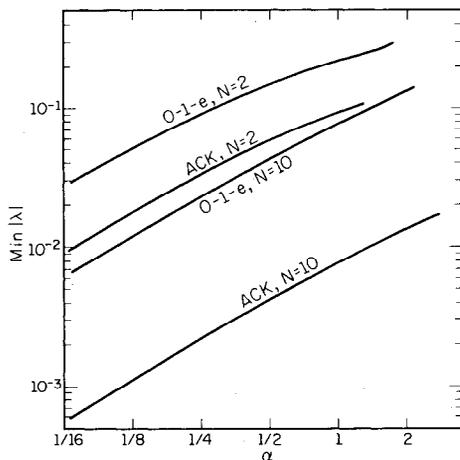


Fig. 3. Magnitude of multiplicity $N - 1$ eigenvalue of F , which serves as ϵ times inverse time constant for convergence of individual transmission probabilities.

curves, we consider the case of ten stations and $\alpha = 0.25$. Fig. 2 indicates that the steady state variance of a given f_i is $\epsilon(0.0016)$ for 0-1-e feedback and $\epsilon(0.23)$ for acknowledgement feedback. From Fig. 3, we see that an approximate "relaxation time" is $1/(0.023)\epsilon \approx 43/\epsilon$ time slots for

0-1-e feedback and $1/(0.0021)\epsilon = 470/\epsilon$ time slots for acknowledgement feedback. We observe here a tradeoff between steady-state accuracy (which is increasing in ϵ) and rate of convergence (which is decreasing in ϵ).

Comparison of Fig. 1 to Figs. 2 and 3 shows a tradeoff between *total* throughput (which is decreasing in α) and steady state accuracy and rate of convergence for the *individual* transmission probabilities (which are both increasing in α).

Note that if one could choose α as a function of N , then for 0-1-e feedback it appears that α should be increasing in N , since for larger N the throughput does not fall off as quickly as α increases. We think it would be desirable to find a strategy such that the optimal α is about the same for all N . We found that changing A_i to $A_i(u) = 0.5\sqrt{u}$ is a step in the right direction—by having a larger derivative of $A_i(u)$ near $u = 0$, we magnify the influence of the terms A_i when the f_i are small, which happens when N is large. However, the difference in performance for $A_i(u) = u$ and $A_i(u) = 0.5\sqrt{u}$ turned out to be rather small, so that alternatives should be sought.

Figs. 2 and 3 indicate, as one would expect, that 0-1-e feedback is much more effective for controlling the individual transmission probabilities f_i than is acknowledgement feedback. The difference in performance is significantly less (but is still large for large N) when we only consider how well the *sum* of transmission rates is controlled. The sum can be written as $e \cdot f$, where e is the vector of all ones. Fig. 4 shows $e^T Q e$ and Fig. 5 shows the magnitude of the eigenvalue of the matrix F for eigenvector e . The first of these approximates $1/\epsilon$ times the steady state variance of $e \cdot f$, and the second gives a speed of convergence measure for $e \cdot f$.

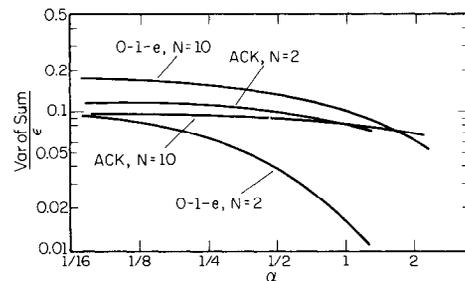


Fig. 4. Scaled asymptotic variance (for ϵ tending to zero) of sum, over all N stations, of individual transmission probabilities, in equilibrium.

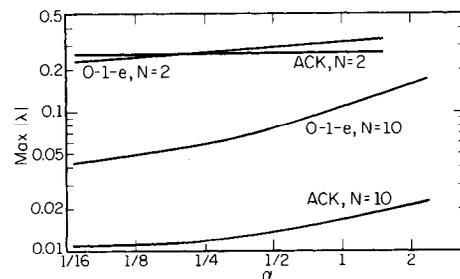


Fig. 5. Magnitude of multiplicity one eigenvalue of F , which corresponds to eigenvector $(1, 1, \dots, 1)^T$ and which serves as ϵ times inverse time constant for convergence of sum of transmission probabilities.

Example 3—Frequency-Hopped Spread Spectrum: Suppose that each time slot is divided into L subslots, and that the frequency band is divided into q sub-bands [7]. Given that a station transmits a packet during a slot, we suppose that it randomly chooses a sequence (called a hopping pattern) of frequency slots to be used during successive subslots. Since error correcting codes will typically be used, some amount of "hitting" is tolerable, although it may be desirable to control the mean traffic level.

In order to obtain feedback information, we assume that during each slot station j chooses a hopping pattern (independently of the one he may be transmitting on) and detects whether or not there is a transmission in each time-frequency subslot. Let $Y_j(k)$ represent the number of subslots in slot k during which station j detected the presence of signals in this chosen hopping pattern. Suppose that a number p^* is determined to be a desirable level of hits. Then we suggest using the transmission strategy (2.1) with

$$G_j(f_j(k), Y_j(k), U_j(k)) = f_j(k) \left\{ - \left(\frac{Y_j(k)}{M} - p \right) - \alpha A_j(f_j(k)) \right\}.$$

For this G , we have

$$\bar{G}_j(f) = f_j \left\{ 1 - \prod_j (1 - f_j/q) - p - \alpha A_j(f_j) \right\}$$

so that \bar{G} has the form (2.6) where Condition C.1) is met. As in the other examples, one method for choosing p would be to consider again a Poisson traffic model; to obtain a mean of λ packets/frequency slot, one should choose $p = 1 - \exp(-\lambda)$. Alternatively, a system designer might prefer direct control of p .

IV. SELECTION OF EQUILIBRIUM POINTS BY OPTIMIZATION

The freedom to choose the functions A_j in our strategies allows the designer to assign priorities to the stations. A possible guideline for choosing these functions is to attempt to minimize some cost function. To accomplish this, we try to force the transmission probability vector f to satisfy the necessary Kuhn–Tucker optimality conditions. We are therefore interested in cases when such conditions completely characterize optimal values of f .

Suppose N stations each operate on a narrowband channel in discrete time. If the transmission probabilities are fixed at f , then the average throughput for station j is

$$S_j = f_j \prod_{i \neq j} (1 - f_i).$$

Suppose we wish to choose f subject to the constraints $0 \leq f \leq \Delta$ in order to minimize the cost $J(f)$ defined by

$$J(f) = \sum_i \psi_i(S_i), \quad (4.1)$$

where the ψ_j are continuously differentiable decreasing functions. The Kuhn–Tucker necessary conditions for opti-

mality of a given f are

$$\frac{\partial J}{\partial f_j} \begin{cases} = 0, & \text{if } 0 < f_j < \Delta_j \\ \leq 0, & \text{if } f_j = \Delta_j \\ \geq 0, & \text{if } f_j = 0. \end{cases}$$

Now

$$\frac{\partial J}{\partial f_j} = \sum_i \psi_i(S_i) \frac{\partial S_i}{\partial f_j}.$$

If $0 < f_j < 1$, then

$$\frac{\partial S_i}{\partial f_j} = \begin{cases} S_j/f_j, & \text{if } i = j \\ -S_i/(1 - f_j), & \text{if } i \neq j \end{cases}$$

and then

$$\begin{aligned} \frac{\partial J}{\partial f_j} &= \left(\frac{1}{f_j} + \frac{1}{1 - f_j} \right) \psi_j(S_j) S_j + \frac{1}{1 - f_j} \sum_i -\psi_i(S_i) S_i \\ &= \frac{1}{1 - f_j} \left\{ -U(S_j)/f_j + \sum_i U(S_i) \right\}, \end{aligned}$$

where, for each i , U_i is the nonnegative function defined by

$$U_i(s) = -\psi_i(s)s. \quad (4.2)$$

Thus, if $0 < f_j < 1$, then

$$\frac{\partial J}{\partial f_j} = 0 \Leftrightarrow f_j = U_j(S_j) / \left(\sum_i U_i(S_i) \right).$$

Treating the cases $f_j = 0$ and $f_j = \Delta_j$ separately, we see that a necessary condition for f to minimize $J(f)$ is that

$$f_j = \left[U_j(S_j) / \sum_i U_i(S_i) \right]_0^{\Delta_j}$$

for all j such that the ratio in square brackets is well-defined extended-real-valued. If $\Delta_j = 1$ for all j , these conditions become

$$\begin{aligned} f_j &= c U_j(S_j) \\ \sum_j f_j &= 1 \end{aligned}$$

for some constant c not depending on j . This suggests, for example, use of the following variation of the transmission strategy of Example 3.1.

$$\begin{aligned} \hat{S}_j(k+1) &= \hat{S}_j(k) + \epsilon [I\{j \text{ successful in slot } k\} - \hat{S}_j(k)] \\ f_j(k+1) &= f_j(k) + \epsilon [c(Y_j(k)) - \bar{U}(\hat{S}_j(k))] \end{aligned} \quad (4.3)$$

where \hat{S}_j is an auxiliary variable updated by station j , $I\{\cdot\}$ denotes the indicator function of the event $\{\cdot\}$, and $\bar{U}_j = \sigma \circ U_j$ for some bounded, monotonically increasing function σ . Recursions of the form (4.3) lead to ordinary differential equation limits which may have more than one locally stable equilibrium point. For example, when $\psi_j(S_j) = 1 - S_j$, J is minimized when the total throughput is maximized and the ODE limit will often have N stable equilibrium points corresponding to $f_j = 1$ for some j and $f_j = 0$ for all other j . The other equilibrium points are not stable. If the cost is changed slightly to

$$\psi_j(S_j) = 1 - (1 + \delta_j) S_j$$

where δ_j are small distinct constants, then the same N points will be asymptotically stable, while only one of them will correspond to a global minimum.

Our next example is motivated by the following fact. Consider a single-server queue modeled as a discrete time Markov process. Customers arrive in time slots according to a Bernoulli process of rate λ and, if there are any customers in the system at the beginning of a time-slot (i.e., arrivals within the slot are not eligible), then one of them is served during the slot with probability s . Then the mean number in the queuing system at the end of a slot when the system is in equilibrium is $\lambda(1 - \lambda)/(s - \lambda)$. Thus, if we let

$$\psi_j(s) = \begin{cases} \frac{w_j}{s - \lambda_j}, & \text{if } s > \lambda_j \\ +\infty, & \text{if } 0 \leq s \leq \lambda_j, \end{cases} \quad (4.4)$$

then the cost $J(\mathbf{f})$ represents a weighted average of the mean backlog at the N stations, assuming that a fixed vector \mathbf{f} of transmission probabilities is used and assuming that, for each j , station j receives packets according to a Bernoulli process of rate λ_j . Note that, for this result to be valid, station j must transmit something in each slot with probability f_j whether or not it actually has a packet. That is, if station j decides to transmit but does not have an actual packet to transmit, it sends a "dummy packet." If the transmission of dummy packets is suppressed and \mathbf{f} is fixed, then $J(\mathbf{f})$ will represent an upper bound on the average delay. One must keep in mind here that if the transmission of dummy packets is suppressed, then the mean traffic intensity may fluctuate more, and it may thus be more difficult to choose transmission probability vectors dynamically.

Returning to the main theme, one hopes that if the f_j 's vary slowly enough, then $J(\mathbf{f}_k)$ may still reflect the mean channel backlog over short periods of time. It should then be a good idea to try to control the \mathbf{f}_k dynamically to keep $J(\mathbf{f}_k)$ small. This idea is sometimes called the quasistatic optimization philosophy.

One problem with the choice (4.4) for the ψ_j is that there may be no \mathbf{f} for which J is finite. For this reason (and others as we will see), we consider the special case of (4.4) with $\lambda_j = 0$ for all j . Thus, we try to minimize

$$J(\mathbf{f}) = \sum_i \frac{w_i}{S_i}.$$

The Kuhn-Tucker conditions then become

$$f_i \propto \frac{w_i}{S_i} \quad \sum_i f_i = 1.$$

Equivalently, $f_i S_i / w_i$ should be the same for all i . Now

$$f_i S_i / w_i = \frac{f_i^2}{w_i(1 - f_i)} \left\{ \prod_j (1 - f_j) \right\}$$

where the quantity in braces does not depend on i . Thus, the optimality conditions are that

$$U_i(f_i) = \frac{f_i^2}{w_i(1 - f_i)}$$

be the same for all i and that $\sum_i f_i = 1$. Since $U_i(f_i)$ is monotonically increasing in f_i , this implies (cf. proof of Lemma 1) that there is a unique solution to the Kuhn-Tucker conditions. Furthermore, the point can be approximately reached through use of the strategy

$$f_j(k+1) = f_j(k) + \epsilon f_j(k) \{ c(Z(k)) - \alpha \tilde{U}_j(f_j(k)) \}$$

where $\tilde{U}_j = \sigma \circ U_j$ for some monotonically increasing function σ . For example, if $\sigma(u) = u/(1 + u)$, then

$$\tilde{U}_j(f) = \frac{f^2}{f^2 + w_j(1 - \delta)}.$$

This strategy fits into the framework of Section II.

V. A SYMMETRIC UNIDIRECTIONAL RING

In this section, we suppose that there are N communication links (rather than N stations), and that only a subset of the other links may interfere with a given link. We write $i \rightarrow j$ to denote the condition that link i interferes with link j for distinct links i and j . We adopt the convention that $i \rightarrow i$ is always true. If link i is used with probability f_i in slot i independently of all other links, then the throughput for link i is given by

$$S_i = f_i \prod_{\substack{j: j \rightarrow i \\ j \neq i}} (1 - f_j).$$

Arguing as in Section III, it is easy to show that a necessary condition for \mathbf{f} to minimize the cost $J(\mathbf{f})$ defined in (4.1) is that

$$f_j = \left[U_j(S_j) / \sum_{i: j \rightarrow i} U_i(S_i) \right]_0^{\Delta_j} \quad (5.1)$$

for all j such that the ratio in square brackets is well-defined extended-real-valued, where the functions U_i are defined in (4.1).

As we know from the broadcast case considered in Section IV, there may be multiple solutions to the conditions (5.1), some of which do not correspond to global minima of J . The purpose of this section is to offer one example where uniqueness does occur. If uniqueness could be proven for a large class of examples, it might provide a methodology for the control of transmission probabilities in general radio networks.

Suppose that the links are numbered 0 through $M - 1$, and that link 0 is also numbered by M . Suppose that $i \rightarrow j$ if and only if $i = j + 1$ or $i = j$. The throughput equations and optimality conditions become

$$S_i = f_i(1 - f_{i-1}), \quad 1 \leq i \leq M$$

and

$$f_i = [U(S_i)/(U(S_i) + U(S_{i+1}))]_0^1, \quad 0 \leq i \leq M - 1. \quad (5.2)$$

We now further restrict ourselves to consider the cost functions $\psi_j(s) = 1/s$ so that $U_j(s) = 1/s$ for all j .

Proposition 2: There are precisely three solutions for \mathbf{f} in (5.2), namely $(0, 0, \dots, 0)$, $(0.5, 0.5, 0.5, \dots, 0.5)$ and

$(1, \dots, 1)$. The first and last maximize J (they have infinite cost) and the second is a global minimum of J .

Proof: Suppose that f satisfies (5.2). If $U(S_{i+1}) < U(S_i) = +\infty$ for some i , then (5.2) implies that $f_i = 1$ so that $S_{i+1} = 0$, which is a contradiction. Hence, if $U(S_i) = +\infty$ for some i , then $U(S_{i+1})$ is also $+\infty$ so that in fact $U(S_j) = +\infty$ for all j . Therefore, the only solutions to (5.2) with $U(S_i) = +\infty$ for some i are those for which $S_i = 0$ for all i . These are the solutions $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$. It remains to show that $(0.5, 0.5, \dots, 0.5)$ is the only solution to (5.2) among vectors f with $0 < f_i < 1$ for all i .

For such f , (5.2) becomes

$$f_i = M(f_{i-1}, f_{i+1})$$

where

$$M(x, y) = \left(1 + \sqrt{\frac{1-x}{y}}\right)^{-1}.$$

Let f^* be any solution to (5.2) with $0 < f_j^* < 1$, and define two sequences $(f^{(n)})$ and $(g^{(n)})$ by

$$f^{(0)} = (\epsilon, \epsilon, \dots, \epsilon)$$

$$g^{(0)} = (1 - \epsilon, 1 - \epsilon, \dots, 1 - \epsilon)$$

and for $n \geq 0$,

$$f_i^{(n+1)} = M(f_{i-1}^{(n)}, f_{i+1}^{(n)}),$$

$$g_i^{(n+1)} = M(g_{i-1}^{(n)}, g_{i+1}^{(n)}).$$

If ϵ is sufficiently small, then

$$(f^{(0)} \leq f^{(1)}), \quad (f^{(0)} \leq f^* \leq g^{(0)}), \quad \text{and} \quad (g^{(1)} \leq g^{(0)}).$$

It follows by induction and the fact that $M(x, y)$ is monotonically increasing in x and in y that

$$f^{(n)} \leq f^{(n+1)} \leq f^* \leq g^{(n+1)} \leq g^{(n)}.$$

Also, $f^{(n)}$ and $g^{(n)}$, for all n , each have all equal entries. Hence, $f^{(n)}$ and $g^{(n)}$ converge to respective vectors f^∞ and g^∞ such that f^∞ and g^∞ each have all entries equal and

$$f^\infty \leq f^* \leq g^\infty.$$

Since $u = 0.5$ is the only solution in $(0, 1)$ to the equation $u = M(u, u)$, it must be that f^∞ , g^∞ , and f^* are all equal.

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