

# Minimum Mean Hitting Times of Brownian Motion with Constrained Drift

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## Abstract

Brownian motion with state-dependent drift in a disk is considered, with reflection at the boundary and initial state uniformly distributed over the disk. The mean time to hit the center of the disk depends on the drift field. The problem considered is to minimize the mean time to hit the center, or a small disk around it, subject to a bound on the integral of the magnitude of the drift vector field. The problem is inspired by the problem of placing a limited amount of routing information in an enormous packet communication network, in order to minimize the mean time to deliver a packet. A related problem is to minimize the mean time to exit a square.

It is demonstrated that angularly symmetric drift fields, with drift pointing directly towards the origin, are not always optimal. If the radius of the disk and the constraint on the total integral of drift are sufficiently large, then a strictly smaller mean hitting time is given by a drift field with a bicycle wheel structure, with strong drift concentrated on the spokes of the wheel.

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# 1 Motivation and Results

Consider a particle that wanders about within the disk  $D$  of radius 1 centered at the origin in the plane, seeking to reach a smaller disk  $A$  with radius  $a$ . For simplicity assume that the smaller disk is also centered at the origin. Suppose there are three aspects to the dynamics of the particle. First, there is a component of Brownian motion—moving the particle about quickly but with no drift. Secondly, there is a component of drift. The drift field is given by  $(b(x) : x \in D)$ , such that  $b(x)$  is a two-vector for each  $x \in D$ . Finally, suppose that whenever the particle would otherwise wander out of the disk  $D$ , it is pushed back into  $D$ , the pushing occurring normal to the boundary of  $D$ . The initial position of the particle is uniformly distributed over  $D$ . Our focus is on  $E[\tau_A]$ , where  $\tau_A$  is the first time that the particle is in  $A$ . Clearly the mean hitting time is a function of the vector field  $b$ . A vector field with a large magnitude can quickly guide the particle to hit  $A$ . The focus of this paper is to examine  $E[\tau_A]$  subject to a constraint on the vector field  $b$ .

For example, the disk could represent a network of millions of small nodes, with each node in radio contact with a few nearby nodes. The particle could represent a packet of information to be routed to some destination region. Most nodes do not know where the destination of the packet is, so those nodes forward the packet in a random direction, causing the Brownian motion component of the dynamics. A small fraction of the nodes do have knowledge about the location of the destination—each of those nodes forwards the packet in a particular direction, causing the drift in the packet dynamics.

For another example, the disk could represent a room and the particle could represent an ant. The inner disk could represent a source of sugar, which the ant is seeking to find. The ant wanders around at random, but occasionally finds a marker pointing in a particular direction, causing some drift.

The location of the particle as a function of time,  $X = (X(t) : t \geq 0)$ , is modeled mathematically as follows. Let  $b$  be a bounded Borel measurable function from  $D$  to  $R^2$ , let  $x \in D$ , let  $(\Omega, \mathcal{F}, P)$  be a probability space with a right-continuous filtration of  $\sigma$ -algebras  $(\mathcal{F}_t : t \geq 0)$ , let  $B$  denote a Brownian motion relative to this space and filtration, let  $\phi$  denote a sample continuous, nondecreasing adapted process with  $\phi(0) = 0$ , and let  $X$  be a measurable adapted process such that the following conditions hold.

$$X(t) = x + B(t) + \int_0^t b(X(s))ds + \int_0^t -X(s)d\phi(s) \quad (1)$$

and

$$\int_0^t I_{\{X(s)=1\}}\phi(ds) = \phi(t) \quad (2)$$

**Proposition 1.1** *For each  $x \in D$  a solution  $(\Omega, \mathcal{F}, P)$ ,  $(\mathcal{F}_t : t \geq 0)$ ,  $B, \phi, X$  to the above conditions exists, and moreover the distribution  $P_x$  on  $\mathcal{B}(C([0, \infty), D))$  induced by  $X$  is uniquely determined. The family of measures  $\{P_x\}$  enjoys the strong Markov property.*

Although the proposition falls within standard theory, a proof is sketched in the appendix. The process  $X$  is called a Brownian motion on  $D$  with drift  $b$  and with pushing normal to the boundary. The family  $\{P_x\}$  is called the diffusion measure for Brownian motion on  $D$  with drift  $b$  and with pushing normal to the boundary.

The optimization problem can now be stated as follows. Suppose the initial distribution of the process is the uniform distribution over the disk  $D$ . Two constraints will be placed on the drift vector field  $b$ . The first is that  $b$  be bounded. This assumption is made so that the process  $X$  is well defined. However, since the particular value of the upper bound on the magnitude of  $b$  is arbitrarily large, this constraint is not very restrictive. The other constraint is that  $\|b\|_1 \leq \Gamma$  for some constant  $\Gamma$ , where  $\|b\|_1$  denotes the  $L_1$  norm of  $b$ :

$$\|b\|_1 = \int_D \|b(x)\|dx. \quad (3)$$

Intuitively, the  $L_1$  norm of  $b$  represents the total amount of resource that is placed in  $D$  to help the particle reach the destination set  $A$ .

The focus of this paper is on the quantity

$$V(a, \Gamma) = \inf\{E[\tau_A] : b \text{ bounded and } \|b\|_1 \leq \Gamma\}, \quad (4)$$

and on the vector fields that achieve the infimum.

A linear scaling of time and space can be used to bound the mean hitting time if the radius of the disk were  $R$  and if the diffusion constant of the Brownian motion were  $\sigma^2$  rather than one. Indeed, suppose  $\hat{X}$  is such a process and the integral of the drift field  $\hat{b}$  over the disk of radius  $R$

is  $\hat{\Gamma}$ . Let  $X(t) = \frac{1}{R}\hat{X}(\frac{tR^2}{\sigma^2})$ . Then  $X$  is a Brownian motion on the unit disk  $D$  with drift field  $b$  given by  $b(x) = \frac{R}{\sigma^2}\hat{b}(Rx)$ , so that  $\|b\|_1 = \frac{\Gamma}{R\sigma^2}$ , and if  $\|X\|$  reaches  $a/R$  at time  $\tau$ , then  $\|\hat{X}\|$  reaches  $a$  at time  $\tau\frac{R^2}{\sigma^2}$ . Therefore, the infimum of mean hitting time of the centered disk of radius  $a$  for Brownian motion on a disk of radius  $R$  and drift constraint  $\|\hat{b}\|_1 \leq \hat{\Gamma}$  is  $\frac{R^2}{\sigma^2}V(\frac{a}{R}, \frac{\hat{\Gamma}}{R\sigma^2})$ .

The problem of computing  $V(a, \Gamma)$ , even numerically, appears formidable. An indication of the difficulty is that  $E[\tau_A]$  is not a convex function of  $b$ . The focus in this paper is limited to analysis of the asymptotics of  $V(a, \Gamma)$  as  $a$  and  $\Gamma$  tend to zero, or as  $\Gamma$  tends to infinity.

A natural class of drift fields to consider are those that are angularly symmetric. Write  $V_{as}(a, \Gamma)$  for the function  $V$  defined above, but under the additional assumption that the drift field  $b$  is angularly symmetric. For such drift fields the hitting problem considered reduces to a hitting problem for the radial component of the diffusion, which itself is a one-dimensional diffusion. It turns out that the optimization problem for one-sided exit of a one-dimensional diffusion is convex, considerably simplifying the analysis of  $V_{as}$ .

**Proposition 1.2** *There are positive constants  $c_1, c_2, c_3$  and  $c_4$  so that the following hold:*

$$c_1 \leq \frac{V_{as}(a, \Gamma)}{-\log(a + \Gamma)} \leq c_2 \quad \text{if } a + \Gamma \leq 1 \quad (5)$$

and

$$\frac{c_3}{\Gamma} \leq V_{as}(a, \Gamma) \leq \frac{c_4}{\Gamma} \quad \text{if } \Gamma \text{ is sufficiently large, and } a \leq 0.5. \quad (6)$$

Equation (5) identifies a logarithmic singularity in  $V_{as}(a, \Gamma)$  as  $a + \Gamma \rightarrow 0$ . On the other hand, (5) reflects the fact that  $V(a, \Gamma)$  tends to zero like  $1/\Gamma$  as  $\Gamma \rightarrow \infty$ , the result not being sensitive to the value of  $a$ .

Of course,  $V(a, \Gamma) \leq V_{as}(a, \Gamma)$ . Are angularly symmetric drift fields optimal for the general problem? That is, does  $V = V_{as}$ ? The answer is no, for the following proposition demonstrates that  $V(a, \Gamma)$  tends to zero nearly as fast as  $1/\Gamma^2$ .

**Proposition 1.3** *There is a constant  $c_5$  so that if  $\Gamma$  is sufficiently large then*

$$V(a, \Gamma) \leq \frac{c_5 \log \Gamma}{\Gamma^2} \quad \text{for } 0 < a < 1 \quad (7)$$

The proof of Proposition 1.3 is based on drift vector fields structured like a bicycle wheel. For such a vector field, the particle moves with drift proportional to  $\Gamma$  until it reaches a spoke. Along a spoke there is a strong drift to quickly move the particle along the spoke towards the center of  $D$ .

A related problem is the problem of exit from a unit square with drift field  $b$  and initial distribution uniform over the square. Let  $U(\Gamma)$  denote the infimum of mean exit times from a unit square, for a Brownian motion with bounded drift field  $b$  such that the initial distribution of the process is uniformly distributed over the square, and  $\|b\|_1 \leq \Gamma$ . As before, while  $b$  is assumed bounded, the numerical value of the bound is not constrained. A relation between  $U$  and  $V$  is obtained as follows. The linear scaling, just as mentioned above for a process in a disk, indicates that the infimum of mean time to exit from a square of side  $\rho$  with the drift field constraint  $\|b\|_1 \leq \Gamma$  is  $\rho^2 U(\frac{\Gamma}{\rho})$ . A square with sides of length  $1/2$  fits within  $D - A$  if  $0 \leq a \leq \frac{1}{2}$ . The Brownian motion on the disk  $D$  starts inside such a square with probability  $\frac{1}{4\pi}$ , and on this event the process must exit the square before it can hit  $A$ . Also, given the initial point is in the square, its conditional distribution is uniform over the square. Therefore, the following inequality obtains.

$$V(a, \Gamma) \geq \frac{1}{16\pi} U(2\Gamma) \quad \text{if } a \leq \frac{1}{2} \quad (8)$$

**Proposition 1.4** *For any  $\delta > 0$  there is a constant  $c_5$  so that*

$$U(\Gamma) \leq \frac{c_6 \log \Gamma}{\Gamma^2} \quad (9)$$

*for all  $\Gamma > 0$ . Furthermore, either  $U(\Gamma) = 0$  for all  $\Gamma > 0$ , or there is a constant  $c_6 > 0$  so that*

$$U(\Gamma) \geq \frac{c_7}{1 + \Gamma^2} \quad (10)$$

*for all  $\Gamma$ .*

We strongly believe the following conjecture is true, but have not found a proof.

**Conjecture 1.1** *For some  $\Gamma > 0$ ,  $U(\Gamma) > 0$ .*

Should the conjecture be proved true, it would show that both  $V(a, \Gamma)$  and  $U(\Gamma)$  tend to zero no more quickly than  $\frac{1}{\Gamma^2}$ , and, as shown by Propositions 1.3 and 1.4, no more slowly than  $\frac{\log \Gamma}{\Gamma^2}$ .

The remainder of this paper is organized as follows. Section 2 considers the analogous problem in one dimension. The first half of the Section concerns the problem of minimizing the mean time to reach an endpoint of an interval, a problem solved by S. Lee [6]. The second half of the section concerns lower bounds on mean and transient exit probability. Both halves of the section are useful for the two dimensional problem. Section 3 treats angularly symmetric drift, establishing Proposition 1.2. Section 4 focuses on a drift field with the wheel structure, establishing Proposition 1.3. Section 5 focuses on the exit problem on a square, establishing Proposition 1.4. Some discussion is given in Section 6. This includes evidence in favor of Conjecture 1.1, a comment on compactification of the optimization problem addressed, and a comment on how the basic questions of this paper relate to a discrete time formulation.

## 2 Hitting Problems in One Dimension

In this section, hitting problems analogous to that considered for the disk are considered for a line segment. The one-dimensional problem is considerably easier, and its solution sheds light on the two dimensional problem. The setup of this section (with the role of 0 and 1 reversed and the constant  $\Gamma$  always equal to one) and the results on minimizing the mean hitting time are due to S. Lee [6]. Consider Brownian motion on the interval  $[0, 1]$  with drift field  $m$ , which is assumed to be a bounded Borel measurable function. The boundary at 1 is reflecting and to be definite suppose state 0 is absorbing. Let  $\tau$  denote the hitting time of state 0. The process  $(X(t) : t \geq 0)$  satisfies a stochastic differential equation similar to (1)-(2). Define for  $0 \leq r \leq 1$ ,

$$h(r) = 2 \int_0^r \int_x^1 \exp(2 \int_x^y m(v) dv) dy dx \quad (11)$$

It is shown that  $h(x) = E_r[\tau]$  as follows (or see [6]). Let  $M(t) = h(X(t)) - t$ . The first derivative of  $h$  is absolutely continuous and its derivative  $h''$  satisfies  $\frac{1}{2}h'' + mh' = -1$  almost everywhere. Thus, the extended version of Ito's formula [5, Theorem 3.7.1] can be applied to yield that  $M$  is a martingale, so that  $h(r) = E_r[h(X(t \wedge \tau))] - E[t \wedge \tau]$  for all  $t \geq 0$ . The notation  $a \wedge b$  denotes the minimum of  $a$  and  $b$ . Taking  $t \rightarrow \infty$  and using the monotone convergence theorem shows that  $E_r[\tau] < \infty$ . In particular,  $h(X(t))$  converges to zero with probability one as  $t \rightarrow \infty$ . Since  $h$  is bounded, continuous and  $h(0) = 0$ ,  $E_r[h(X(t \wedge \tau))]$  also converges to zero. Thus,  $h(r) = E_r[\tau]$  as claimed.

For fixed  $r$ ,  $h(r)$  is a strictly convex function of  $m$ , and the set of bounded measurable  $m$  with  $\|m\|_1 \leq \Gamma$  is convex. To ensure that  $m$  minimizing or maximizing  $h(r)$  exists, the following compactification of this set is considered. Let  $\mu$  denote the signed measure on  $[0, 1]$  with density  $m$ . Then  $h(x)$  can be expressed as

$$h(u) = 2 \int_0^u \int_x^1 \exp(2\mu([x, y])) dy dx. \quad (12)$$

Equation (12) can be used to define  $h(x)$  for any signed measure  $\mu$  with finite total variation, denoted by  $\|\mu\|$ . The set of finite measures with total variation less than or equal to  $\Gamma$  under the weak topology is compact,  $h(r)$  is a continuous, strictly convex function on this set, and the subset of this set having bounded densities is dense in the set [6].

The compactification can be simply viewed as a method for finding the infimum or supremum of  $h(r)$  over all bounded measurable functions  $m$ . Alternatively, a more general notion of Brownian motion with drift can be considered, following LeGall [7], as now briefly described. The measure  $\mu$  can be viewed as a (one-dimensional vector)-valued measure. A Brownian motion with drift field  $\mu$  can be defined as a process  $X$  satisfying

$$X(t) = x + B(t \wedge \tau) + \int_{-\infty}^{\infty} \Lambda(X, a, t \wedge \tau) \mu(da) + \phi(t) \quad (13)$$

in which  $\phi$  satisfies a condition as in (2), and  $\Lambda(X, a, t)$  is the local time of  $X$  at  $a$  up to time  $t$  (see [5, Section 3.6]). Existence and uniqueness of solutions in distribution can be established by scaling space to remove the drift, as in [5, Section 5.5.B]. A point mass in  $\mu$ , or equivalently a delta function term in  $m$ , gives rise to the behavior demonstrated by skew Brownian motion [6].

## 2.1 Minimizing the mean hitting time for one dimension

This section summarizes results of S. Lee [6]. Lee noted that the calculus of variations can be applied to the problem, and the results are rederived here using that method. The only new contribution of the subsection is the intuitive explanation of the the solution. Fix  $r \in [0, 1]$  and  $\Gamma \geq 0$ . Consider the problem of finding the infimum of  $h(r)$  over all  $m$  with  $\|m\|_1 \leq \Gamma$ . Equivalently, consider the problem of minimizing  $h(r)$  over all signed measures  $\mu$  with  $\|\mu\| \leq \Gamma$ . Restrict attention, without loss of optimality, to nonpositive measures  $\mu$ . The optimality conditions are that there exists a

positive constant  $\lambda$  such that

$$\int_0^{r \wedge u} \int_u^1 \exp(2\mu([x, y])) dy dx \leq \frac{\lambda}{4} \quad (14)$$

for  $0 \leq u \leq 1$ , and equality holds in (14) whenever  $u$  is in the support of  $\mu$ . Setting  $\Phi(x) = \exp(-2\mu([0, x]))$ , (14) can be rewritten as

$$\int_0^{r \wedge u} \frac{1}{\Phi(x)} dx \int_u^1 \Phi(y) dy \leq \frac{\lambda}{4} \quad (15)$$

Note that the left side of (15) is strictly decreasing in  $u$  over the interval  $[r, 1]$ . Therefore,  $\mu$  is supported entirely on the interval  $[0, r]$ . In some cases, however,  $\mu$  has an atom of mass at  $r$  (which can be conveniently expressed by allowing  $m$  to have a delta function translated to  $r$ ), which tends to steer the particle to the left of  $r$ . Note also the the left side of (15) is continuous in  $u$ , strictly positive on the interior of  $[0, 1]$ , and with limit zero at the endpoints of  $[0, 1]$ . Consequently the optimal drift field is zero in a neighborhood of each endpoint (even if  $r = 1$ ). Intuitively, the drift near 1 is zero because the reflection at 1 pushes the particle down enough in this region, so that the drift can be better used elsewhere. Intuitively, the drift is zero near zero because, since the particle is absorbed at zero, it spends less time near zero than elsewhere. So again the drift is better used elsewhere. The optimal drift function  $m$  is specified in the following proposition. The notation  $I_{[x, y]}$  is used to denote the indicator function of the interval  $[x, y]$ , and  $\delta_x$  is used to indicate a delta function at  $x$ .

**Proposition 2.1** (*S. Lee [6]*) *Given  $0 < r \leq 1$  and  $\Gamma \geq 0$ , the drift field  $m$  on  $[0, 1]$  that minimizes  $E_r[\tau]$  for Brownian motion with drift  $m$  and reflection at 1, is given as follows.*

Case i: *If  $0 < r \leq \frac{1}{1+e^{2\Gamma}}$  then  $-m = \Gamma\delta_r$ . This yields  $\min E_r[\tau] = r^2 + 2r(1-r)e^{-2\Gamma}$ .*

Case ii: *If  $\frac{1}{1+e^{2\Gamma}} \leq r \leq \frac{2\Gamma+1}{2(1+\Gamma)}$  then  $-m = aI_{[\gamma, r]} + d\delta_r$ , where  $\gamma$  denotes the unique positive solution to*

$$\gamma \exp(2\Gamma + 1 - \frac{r}{\gamma}) = 1 - r, \quad (16)$$

*$a = \frac{1}{2\gamma}$ , and  $d = \Gamma + \frac{1}{2} - \frac{r}{2\gamma}$ . This yields  $\min E_r[\tau] = 2r\gamma + \gamma^2$ .*

Case iii: *If  $\frac{2\Gamma+1}{2(1+\Gamma)} \leq r \leq 1$  then  $-m = (1 + \Gamma)I_{[\gamma, 1-\gamma]}$ , where  $\gamma = 1/(2 + 2\Gamma)$ . This yields  $\min E_r[\tau] = \frac{1}{1+\Gamma} - (1-r)^2$ .*



**Proof.** It is straightforward to check that the  $m$  specified satisfies the optimality conditions. Or, see the proof of [6]. ■

See Figure 1 for an illustration. The case the particle begins at the right endpoint may

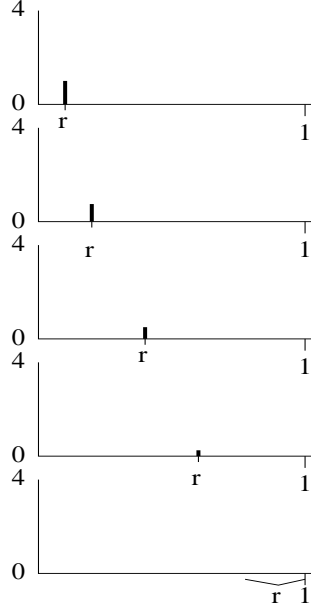


Figure 1: Sketch of  $-m$  for the drift function  $m$  on  $[0, 1]$  that minimizes the time to reach 0 from initial point  $r$ , for  $\Gamma = 1$  and initial point  $r = 0.1, 0.2, 0.4, 0.6$ , and  $0.75 \leq r \leq 1$ , respectively.

be the most important in applications. Therefore, that case is restated with general scaling, as a corollary.

**Corollary 2.1** (*S. Lee [6]*) *Let  $L > 0$ ,  $\Gamma \geq 0$  and  $\sigma^2 \geq 0$ , and let  $m$  be a measurable function on  $[0, L]$  with  $\|m\|_1 \leq \Gamma$ . Let  $X$  be a process on  $[0, L]$  such that for a nondecreasing, continuous process  $\phi$  with  $\phi(0) = 0$ :*

$$X(t) = L + \sigma B(t \wedge \tau) + \int_0^{t \wedge \tau} m(t)dt - \phi(t), \quad (17)$$

$\phi$  increases only on the set  $X(t) = L$ , and  $\tau$  is the time  $X$  first hits zero. Then

$$E[\tau] \geq \frac{L^2}{\sigma^2 + \Gamma}. \quad (18)$$

Equality holds in (18) if and only if  $m = -\frac{\Gamma + \sigma^2}{L} I_{[\gamma, 1-\gamma]}$  (up to a set of measure zero), where

$$\gamma = \frac{L\sigma^2}{2\sigma^2 + 2\Gamma}.$$

**Proof.** If  $\sigma > 0$ , the Corollary follows by applying Proposition 2.1 for  $r = 1$  to the process  $Z(t) = X(tL^2/\sigma^2)/L$ . If  $\sigma = 0$  then Jensen's inequality shows the optimality of constant velocity.

■

To better appreciate Corollary 2.1, consider the effect of drift and diffusion separately. If  $\Gamma = 0$  then  $X$  relies on its diffusion component to hit zero, resulting in the mean hitting time  $L^2/\sigma^2$ . On the other hand, if  $\Gamma > 0$  then a constant vector field  $m(x) = -\Gamma/L$  yields  $E[\tau] \leq L^2/\Gamma$ , no matter what the value of  $\sigma^2$ . Thus, in general, the mean hitting time is less than or equal to  $\min\{L^2/\sigma^2, L^2/\Gamma\}$ . Corollary 2.1 shows that when both drift and diffusion are present, then it is precisely the sum  $\sigma^2 + \Gamma$ , that determines their joint effectiveness in moving the particle to zero, if the drift field has the optimal shape.

## 2.2 Maximizing the mean hitting time for one dimension

The previous subsection concentrates on drift fields that minimize the mean time to hit zero. This subsection focuses on maximizing the mean time to hit zero. For simplicity, only the case of initial condition  $r = 1$  is considered in this subsection.

Since  $h(1)$  is a convex, continuous function defined on a compact, convex set of signed measures  $\{\mu : \|\mu\| \leq \Gamma\}$ , it follows that it is maximized at the extreme points of this set. The extreme points of the set are those measures supported by a single point. Maximizing over the location of the point, yields that the maximizing drift field is the one with all mass at the midpoint of the interval:  $m^{\max}(x) = \Gamma\delta_{1/2}$ , yielding

$$E_1^{\max}[\tau] = \frac{1 + e^{2\Gamma}}{2} \quad (19)$$

It turns out that as  $\Gamma \rightarrow \infty$ ,  $E_1^{\max}[\tau]$  is logarithmically equivalent to  $e^{2\Gamma}$  for any sequence of nonnegative measures  $\mu_\Gamma$ , as long as not too much mass is placed near the endpoints of the interval  $[0, 1]$ . This can be seen directly from the expression (12) for  $h(1)$ . It can also be deduced from the theory of large deviations, as shown next.

The following lemma provides an estimate on the distribution of the exit time, rather than just on the mean of the exit time. It is used in Section 4.

**Lemma 2.1** *Let  $\tilde{m}$  be a nonnegative continuous function on  $[0, 1]$  with  $\|\tilde{m}\|_1 = 1$  and let  $m$  be defined on  $[0, L]$  by  $m(x) = \Gamma\tilde{m}(x/L)/L$ . (Note that  $\|m\|_1 = \Gamma$ ). Consider a Brownian motion on  $[0, L]$  with drift field  $m$ , with reflection at  $L$ , and with initial state  $L$ . For any  $\epsilon > 0$  there exists a constant  $c_\epsilon$  (depending on  $\tilde{m}$  but not on  $L$ ) so that if  $\Gamma \geq c_\epsilon$  then*

$$P[\tau \geq L^2 \exp((2 - \epsilon)\Gamma)] \geq 1 - \epsilon. \quad (20)$$

**Proof.** The process  $Z(t) = X(tL^2/\Gamma)/L$  is a scaled Brownian motion on  $[0, 1]$  with drift  $\tilde{m}$ , diffusion term  $1/\Gamma$ , and it hits zero at time  $\tilde{\tau} = \tau\Gamma/L^2$ . Furthermore, (20) is equivalent to

$$P[\tilde{\tau} \geq \Gamma \exp((2 - \epsilon)\Gamma)] \geq 1 - \epsilon. \quad (21)$$

In particular, the parameter  $L$  is scaled out. The theory of large deviations (specifically, Lemma 4.4.2 of Freidlin and Wentzell [3] with the diffusion constant  $\epsilon^2$  of [3] set to  $\frac{1}{\Gamma}$ ) will be applied to the process  $Z$ . Accordingly, define a potential  $V_0$  by

$$V_0 = \inf_{\phi} \frac{1}{2} \int_0^T (\tilde{m}(\phi(t)) - \dot{\phi}(t))^2 dt \quad (22)$$

where the infimum is over all  $T > 0$  and all absolutely continuous functions  $\phi$  with range  $[0, 1]$  with  $\phi(0) = 1$  and  $\phi(T) = 0$ . A time scaling (see [3, Lemma 4.3.1]) shows that the function  $\phi$  that achieves the infimum in the definition of  $V_0$  satisfies  $\dot{\phi}(t) = -\tilde{m}(\phi(t))$ , from which it follows that that  $V_0 = 2 \int_0^1 \tilde{m}(x) dx = 2$ . Theorem 4.4.2 of [3] implies that

$$\lim_{\Gamma \rightarrow \infty} P[\tilde{\tau} \geq \exp((2 - \epsilon/2)\Gamma)] = 1, \quad (23)$$

which in turn implies (21) for sufficiently large  $\Gamma$ . ■

The intuition behind the lemma is that if  $\Gamma$  is large, then the probability the process hits 0 during a given short interval is very small—requiring a “large deviation” of the process. Since the process frequently comes back to  $L$ , effectively restarting,  $\tau/E[\tau]$  is asymptotically exponentially distributed.

### 3 Angularly Symmetric Drift on the Disk

Proposition 1.2 is proved in this section. If the drift field  $b$  is angularly symmetric, for the purposes of minimizing the hitting time of  $A$ , the drift can be taken to be pointing directly at the origin.

Thus, it is assumed throughout this section that  $b(x) = \frac{x}{\|x\|} b_o(\|x\|)$  for  $x \neq 0$ , where  $b_o(v) \leq 0$  for  $0 \leq v \leq 1$ . The constraint  $\|b\|_1 \leq \Gamma$  becomes  $\int_0^1 |b_o(v)| 2\pi v dv \leq \Gamma$ . By Ito's formula, the radial part of  $X(t)$ , denoted  $R(t) = \|X(t)\|$ , is a one-dimensional Brownian motion on the interval  $[0, 1]$  with reflection at 1 and drift field  $m$  given by

$$m(x) = \frac{1}{2x} + b_o(x). \quad (24)$$

Let  $V_{as}^\partial(a, \Gamma)$  be defined the same way as  $V_{as}(a, \Gamma)$ , but with the initial state of the process  $X$  taken to be uniformly distributed on the boundary of the unit disk  $D$ , rather than on all of  $D$ . The following relation between  $V$  and  $V^\partial$  will be proved for  $0 < a < s < 1$ .

$$(1 - s^2)s^2 V_{as}^\partial\left(\frac{a}{s}, \frac{\Gamma}{s}\right) \leq V_{as}(a, \Gamma) \leq V^\partial(A, \Gamma) \quad (25)$$

Let  $X$  denote a Brownian motion with drift field  $b$  satisfying  $\|b\|_1 \leq \Gamma$  and with initial distribution uniform in  $D$ . The event  $\|X(0)\| > s$  has probability  $1 - s^2$  and on the event the process must reach the centered disk  $D_s$  of radius  $s$  before reaching  $A$ . Once the boundary of  $D_s$  is reached, the additional time needed to reach  $A$  is stochastically larger than the time to hit  $A$  starting from the boundary of  $D_s$  with the boundary of  $D_s$  made reflecting inward. By linear scaling in space and time, the mean time to reach  $A$  from the boundary of  $D_s$  is thus at least  $s^2 V_{as}^\partial\left(\frac{a}{s}, \frac{\Gamma}{s}\right)$ . This establishes the first inequality in (25). The second inequality is obvious.

By (25) it suffices to prove Proposition 1.2 for  $V_{as}$  replaced by  $V_{as}^\partial$ . Therefore, throughout the remainder of this section, assume that  $X(0)$  is uniformly distributed on the boundary of  $D$ .

Two upper bounds on  $V_{as}^\partial$  are obtained by simply considering particular choices for  $b_o$ . Equation 11 translated for the interval  $[a, 1]$  and initial state  $r = 1$  yields the following expression for  $E[\tau_A]$ .

$$E[\tau_A] = 2 \int_a^1 \int_x^1 \exp\left(2 \int_x^y m(v) dv\right) dy dx \quad (26)$$

For the first upper bound, the vector field  $b_o(v)$  is selected to make  $m(v)$  zero in an interval with left endpoint  $a$ . Specifically, assume that  $a + \Gamma\pi \leq 1$ , and define

$$b_o(v) = \begin{cases} -\frac{1}{2v} & \text{if } a \leq v \leq \frac{\Gamma}{\pi} \\ 0 & \text{else} \end{cases} \quad (27)$$

Then  $\|b\|_1 = \Gamma$ , and (26) yields

$$E[\tau_A] = -\log\left(a + \frac{\Gamma}{\pi}\right) + \frac{a^2 - 1}{2} + \frac{\Gamma^2}{2\pi^2} + \frac{\Gamma}{a\pi + \Gamma} \quad (28)$$

Therefore,

$$V_{as}^\partial(a, \Gamma) \leq -\log\left(a + \frac{\Gamma}{\pi}\right) + \frac{3}{2} \quad \text{if } a + \frac{\Gamma}{\pi} \leq 1 \quad (29)$$

This establishes the second equality of (5).

For the second upper bound, suppose that  $\Gamma > \pi$  and let  $b_o(v) = -\frac{1}{2v} - f$  where  $f$  is selected so that  $\|b\|_1 = \Gamma$ , namely  $f = \frac{\Gamma - \pi}{\pi}$ . Consequently  $m(v) = -f$ , so that  $E[\tau_A] \leq \frac{1}{f}$ . Therefore,

$$V_{as}^\partial(a, \Gamma) \leq \frac{\pi}{\Gamma - \pi} \quad \text{if } \Gamma > \pi \quad (30)$$

This establishes the second equality of (6).

The following lemma is used for the proof of the first equality of (5).

**Lemma 3.1** *Let  $0 < a < s$  and  $\Gamma > 0$  such that  $a + \Gamma \leq s$ . Then*

$$\inf_f \int_a^s \exp\left(\int_x^s \frac{1 - f(v)}{v} dv\right) dx = s\left(\log\left(\frac{s}{a + \Gamma}\right) + \frac{\Gamma}{a + \Gamma}\right), \quad (31)$$

where the infimum is over all bounded, measurable functions  $f$  on the interval  $[a, s]$  with  $\|f\|_1 = \Gamma$ .

The function to be minimized is convex in  $f$  and the local optimality conditions are the following. There exists  $\lambda > 0$  so that

$$\int_a^u \exp\left(\int_x^s \frac{1 - f(v)}{v} dv\right) \leq \lambda u \quad (32)$$

for  $a \leq u \leq s$ , and equality holds in (32) for  $u$  such that  $f(u) \neq 0$ . It is easy to check that the function  $f = I_{[a, a+\Gamma]}$  satisfies the optimality conditions, and the resulting value of the function to be minimized is given by the right side of (31).

Turn now to the problem of finding a lower bound on  $V_{as}^\partial$  for small  $\Gamma$  and  $a$ . For convenience,  $V_{as}^\partial(a, \pi\Gamma)$  is bounded, rather than  $V_{as}^\partial(a, \pi)$ . Let  $b$  be a bounded measurable drift on the unit disk  $D$  with  $\|b\|_1 \leq \pi\Gamma$ . Fix  $s$  with  $a < s < 1$ . Reducing the region of integration in (26) yields that

$$E[\tau_A] \geq 2 \int_a^s \int_s^1 \exp\left(2 \int_x^y m(v) dv\right) dy dx \quad (33)$$

But for  $s \leq y \leq 1$ ,

$$2 \int_s^y m(u) du \geq \frac{1}{s\pi} \int_s^y b_o(u) 2\pi u du = -\frac{\Gamma}{s} \quad (34)$$

so that

$$\begin{aligned} E[\tau_A] &\geq 2e^{-\frac{\Gamma}{s}} \int_a^s \int_s^1 \exp(2 \int_x^s m(v) dv) dy dx \\ &= 2(1-s)e^{-\frac{\Gamma}{s}} \int_a^s \exp(2 \int_x^s m(v) dv) dx \\ &\geq 2s(1-s)e^{-\frac{\Gamma}{s}} \left( \log\left(\frac{s}{a+\Gamma}\right) + \frac{\Gamma}{a+\Gamma} \right), \end{aligned}$$

where the last inequality follows by Lemma 3.1 with  $f(v) = 2vb_o(v)$ . Consequently,

$$V_{as}^\partial(a, \pi\Gamma) \geq 2s(1-s)e^{-\frac{\Gamma}{s}} \log\left(\frac{s}{a+\Gamma}\right), \quad (35)$$

which establishes the first inequality of (5).

It remains to establish the first inequality of (6) with  $V_{as}$  replaced by  $V_{as}^\partial$ . Suppose that  $0 < a \leq 0.5$ , and use the fact that  $\tau_A$  is greater than or equal to the first time that  $\|X(t)\| \leq 0.5$ . Furthermore, the integral of the negative part of  $m$  over the interval  $[0.5, 1]$  is bounded above as follows.

$$\int_{0.5}^1 |m(v) \wedge 0| dv \leq \int_u^v |b_o(r)| dr \leq \frac{1}{\pi} \int_{0.5}^1 |b_o(v)| 2\pi v dv \leq \frac{\Gamma}{\pi} \quad (36)$$

Thus,  $E[\tau_A]$  is greater than or equal to the mean time to hit the left endpoint for a one dimensional Brownian motion on an interval of length 0.5 with integral of drift field  $\frac{\Gamma}{\pi}$ , with initial point equal to the right endpoint of the interval. Apply Corollary 2.1 to bound this conditional expectation from below, to yield that

$$V_{as}^\partial(a, \Gamma) \geq \frac{(0.5)^2}{1 + \frac{\Gamma}{\pi}}. \quad (37)$$

This proves the first equality of (6), so the proof of Proposition 1.2 is complete.

## 4 Bicycle Wheel Drift on the Disk

Proposition 1.3 is established in this section by consideration of a drift vector fields with the geometry of a bicycle wheel. Such a vector field  $b$  is defined for each  $\Gamma > 0$ . It is shown that

$\|b\|_1 \leq \Gamma$  and that for some constant  $c_5$ , if  $\Gamma$  is sufficiently large then  $E[\tau_{\{0\}}] \leq \frac{c_5 \log \Gamma}{\Gamma^2}$ . Since for given  $\Gamma$ , the vector field  $\|b\|$  is bounded outside any neighborhood of the origin, this will establish Proposition 1.3.

Define the following parameters:

$$\begin{aligned}
M &= \frac{\Gamma}{9 \log \Gamma} && \text{(the number of spokes)} \\
L &= \frac{2 \log \Gamma}{\Gamma^2} && \text{(the one-sided thickness of the spokes)} \\
f &= \Gamma^2 && \text{(drift of radial component within spokes and hubs)} \\
g &= \frac{\Gamma}{10\pi} && \text{(drift magnitude outside spokes and hubs)} \\
\phi &= \frac{2\pi}{2M} && \text{(half the angle between adjacent spokes)} \\
H &= \frac{L}{\sin(\phi)} && \text{(radius of inner hub)}
\end{aligned}$$

The parameter  $M$  is to be an integer.

Throughout this section,  $x$  denotes the rectangular coordinates of a point in  $D$  and  $(r, \theta)$  denotes the polar coordinates of the same point. Assume the vector field  $b$  is invariant under rotations by  $2\phi$ , so that  $b(x)$ , or equivalently  $b((r, \theta))$ , need only be specified for the sector  $|\theta| \leq \phi$ . On that sector,  $b$  is given as follows.

$$b(x) = \begin{cases} -\frac{x}{r}(\frac{1}{2r} + f) & \text{if } 0 < r \leq H \quad \text{(inner hub region, less the origin)} \\ -\frac{x}{r}(\frac{1}{2r} + f) & \text{if } H < r \leq L + H \text{ and } L < |x_2| \leq r \cos \phi \quad \text{(outer hub region)} \\ \left( \begin{array}{c} -\frac{1}{2x_1} - \frac{f(1-|\sin(\theta)|)}{\cos(\theta)} \\ -f \operatorname{sgn}(x_2) \end{array} \right) & \text{if } r > H \text{ and } |x_2| \leq L \quad \text{(first spoke region)} \\ 0 & \text{if } x = 0 \\ \left( \begin{array}{c} 0 \\ -g \operatorname{sgn}(x_2) \end{array} \right) & \text{else} \quad \text{(the complement of spoke and hub regions)} \end{cases} \tag{38}$$

See Figure 2.

The exact choice of the drift field  $b$  was made so that the following two properties hold. The radial component of the drift is  $\frac{x}{r} \cdot b(x) = -(\frac{1}{2r} + f)$  for all  $x$  such that  $b(x) \neq 0$ . Thus,

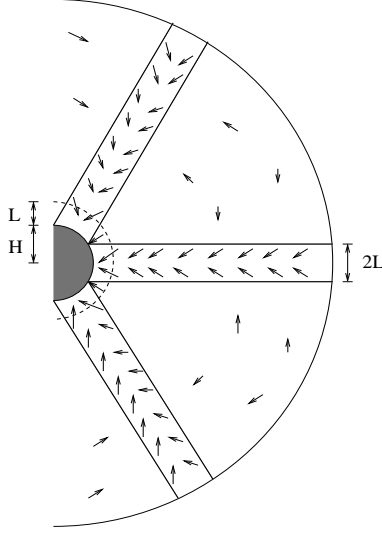


Figure 2: Sketch of  $b$

according to Ito's differential rule, the radial process  $R(t) = \|X(t)\|$  behaves as a Brownian motion with constant drift  $-f$  in the region that  $b(x) \neq 0$ . If  $X(t)$  is in the interior of a spoke region, then its distance from the center of the spoke evolves as a Brownian motion with constant drift  $-f$  and reflection at 0.

The bound claimed for  $\|b\|_1$  is established as follows. Assume  $\Gamma$  is large enough so that  $M \geq 2$ . Then  $\phi \leq \pi/2$  so that  $\phi \geq \sin(\phi) \geq 2\phi/\pi$  which implies that  $LM/\pi \leq H \leq LM/2$ . Also, if  $x$  is in the first spoke region then its angle  $\theta$  satisfies  $|\theta| \leq \phi \leq \pi/2$  so that  $\frac{1-|\sin(\theta)|}{\cos(\theta)} \leq 1$ .

Observe that  $\|b\|_1 \leq I_1 + I_2 + I_3 + I_4$ , where

$$\begin{aligned}
 I_1 &= \text{integral of } b \text{ over inner and outer hub regions} \\
 &\leq 2\pi \int_0^{H+L} \left(\frac{1}{2r} + f\right) r dr \\
 &= \pi(H+L) + \pi f(H+L)^2 \leq \pi\left(\frac{LM}{2} + L\right) + \pi f\left(\frac{LM}{2} + L\right)^2 = O(1),
 \end{aligned}$$

$$\begin{aligned}
 I_2 &= M * (\text{integral of } \frac{1}{2x_1} \text{ over the first spoke}) \\
 &\leq 2ML \int_H^1 \frac{1}{2x_1} dx_1 = ML \log\left(\frac{1}{H}\right) = o(1),
 \end{aligned}$$

$$I_3 = M * (\text{integral of } 2f \text{ over the first spoke})$$



$$\leq 4MLf = \frac{8\Gamma}{9},$$

$$\begin{aligned} I_4 &= (\text{integral of } g \text{ over the complement of spoke and hub regions}) \\ &\leq \pi g = \frac{\Gamma}{10}. \end{aligned}$$

Therefore,  $\|b\|_1 \leq \Gamma$  for sufficiently large  $R$  as claimed.

It remains to establish the bound for  $E[\tau_{\{0\}}]$ . The idea of the proof is to consider a series of epochs. Let the starting point  $x$  be an arbitrary point of  $D$ . Let  $D_1 = \{x : \theta = 2k\phi \text{ for an integer } k\}$ . Let  $\tau_1$  denote the hitting time of  $D_1$  and let  $\tau_2 = \min\{t \geq \tau_1 : R(t) \leq H\}$ . Here  $(R(t), \Theta(t))$  is used for the polar coordinates of  $X(t)$ . Define a process  $\eta = (\eta(t) : t \geq \tau_1)$  as follows.

$$\eta(t) = \begin{cases} L - (\text{distance of } X(t) \text{ from } D_1) & \text{if } \tau_1 \leq t < \tau_2 \\ H + L - R(t) & \text{if } t \geq \tau_2 \end{cases} \quad (39)$$

Let  $\rho$  denote the first time after  $\tau_1$  that  $\eta$  reaches zero. Note that throughout the time interval  $[\tau_1, \rho]$ ,  $X(t)$  is either in a spoke, inner hub or outer hub region. Let the first epoch end at time  $\rho \wedge \tau_{\{0\}}$ . Let  $S$  denote the event that 0 is reached at the end of the first epoch. On the event  $S$ , no more epochs need be defined. On the complement of  $S$ , the second epoch begins at time  $\rho$  with the initial state  $X(\rho)$ , and otherwise it is defined similarly to the first epoch. The series of epochs continues until the origin is reached at the end of an epoch. Let  $C$  be given by

$$C = \frac{4\pi}{2Mg} = \frac{180\pi^2 \log \Gamma}{\Gamma} \quad (40)$$

It will be shown that  $P_x[S] \geq 1/3$  and that  $E_x[\rho \wedge \tau_{\{0\}}] \leq C$ , no matter what the starting point  $x$ . These two facts imply that  $E_x[\tau_{\{0\}}] \leq 3C$ , so that the proof of Proposition 1.3 will be complete.

First, a bound on  $E_x[\tau_1]$  is derived. Let  $\xi(t)$  denote the distance between  $X(t)$  and the closest point of  $D_1$ . The process  $\xi$  takes values in the interval  $[0, \sin(\phi)]$ , and  $\tau_1$  is equal to the first time that  $\xi$  reaches zero. But  $\xi$  is stochastically dominated from above by Brownian motion on the interval  $[0, \sin(\phi)]$ , with drift  $-g$ , reflection at the upper endpoint, and initial state equal to the upper endpoint. Thus,

$$E[\tau_1] \leq \frac{\sin(\phi)}{g} \leq \frac{2\pi}{2Mg} = \frac{C}{2} \quad (41)$$

Next, it is shown that  $\rho$  is large with high probability. We claim that over the interval  $[\tau_1, \rho]$ ,  $\eta$  is stochastically bounded below by a Brownian motion on the interval  $[0, L]$  with drift  $f$ , reflection at each endpoint, and initial state  $L$ . Indeed, over the interval  $[\tau_1, \rho \wedge \tau_2]$  this is exactly the law of  $\eta$ . If  $\rho > \tau_2$ , then at time  $\rho \wedge \tau_2$  the process  $\eta$  jumps back to  $L$ , and over  $[\rho \wedge \tau_2, \rho]$  it is Brownian motion on the interval  $[0, L + H]$  with drift  $f$  and reflection at  $L + H$ . This shows the claim. Let  $0 < \epsilon < 1$ . By Lemma 2.1, there is a constant  $c_\epsilon$  so that if  $fL \geq c_\epsilon$ , then

$$P[\rho - \tau_1 \geq L^2 e^{(2-\epsilon)fL}] \geq \frac{2}{3} \quad (42)$$

By the choice of parameters in this section, if  $\Gamma$  is sufficiently large then  $fL = 2 \log \Gamma \geq c_\epsilon$  and

$$L^2 e^{(2-\epsilon)fL} = \frac{4(\log \Gamma)^2}{\Gamma^{2\epsilon}} > 3C \quad (43)$$

Thus, for sufficiently large  $\Gamma$ ,

$$P[\rho > 3C] \geq \frac{2}{3} \quad (44)$$

Let  $\tilde{R} = (\tilde{R}(t) : t \geq 0)$  denote a sample continuous random process which is identical to  $R(t)$  up to time  $\rho$ . After time  $\rho$ ,  $\tilde{R}$  is constructed to be a Brownian motion on the interval  $[0, \Gamma]$  with constant drift  $-f$  and reflection at the endpoints. Then over the interval  $[\tau_1, \infty)$ , the process  $\tilde{R}$  is a Brownian motion on the interval  $[0, \Gamma]$  with constant drift  $-f$  and reflection at the endpoints. Let  $\tilde{\tau}$  denote the first time after  $\tau_1$  that  $\tilde{R}$  hits zero. Then  $E[\tilde{\tau} - \tau_1] \leq \frac{1}{f} \leq \frac{C}{2}$ . Combining this with (41) shows that  $E[\tilde{\tau}] \leq C$  for  $R$  sufficiently large. Either  $\rho \leq \tilde{\tau}$  or  $\tilde{\tau} = \tau_{\{0\}}$ , so that  $\rho \wedge \tau_{\{0\}} \leq \tilde{\tau}$ . The inequality  $E[\rho \wedge \tau_{\{0\}}] \leq C$  is therefore proved, as desired.

Markov's inequality implies that  $P[\tilde{\tau} \leq 3C] \geq 2/3$ . Thus,

$$P[S] = P[\tilde{\tau} < \rho] \geq P[\{\tilde{\tau} \leq 3C\} \cap \{\rho > 3C\}] \geq P[\tilde{\tau} \leq 3C] + P[\rho > 3C] - 1 \geq 1/3 \quad (45)$$

The proof of Proposition 1.3 is complete.

## 5 Exiting a Square

Proposition 1.4 is established in this section. The upper bound, (9), is immediate from (8) and Proposition 1.3. A scaling argument will be used to establish the lower bound, (10), under the assumption that  $U(2w) > 0$  for some  $w > 0$ .

By its definition,  $U(\Gamma)$  is nonincreasing in  $\Gamma$ . Also,  $U(0) \leq \frac{1}{6}$ , since the mean time to exit from the square is smaller than or equal to the mean time of the first coordinate of the Brownian motion to exit from an interval of length one, and the initial distribution of the first coordinate is uniform over the interval.

Let  $\Gamma > 0$  and  $\epsilon > 0$ , let  $b$  be a bounded vector field on the unit square  $S = [0, 1]^2$  with  $\|b\|_1 = \Gamma$  such that  $E[\tau] \leq U(\Gamma) + \epsilon$ , where  $\tau$  is the exit time from  $S$  of a Brownian motion  $X$  with drift  $b$  on  $S$  and with initial point  $X(0)$  uniformly distributed over  $S$ . Partition  $S$  into four subsquares,  $S_1, S_2, S_3$ , and  $S_4$ , with  $S_1 = [0, 0.5]^2$ . Let  $\Gamma_i = \int_{S_i} \|b(x)\| dx$ , and let  $\tilde{\tau}$  denote the first time that  $X$  reaches the boundary of one or more of the subsquares. Of course  $\tau \geq \tilde{\tau}$ . The idea now is to use the fact that on the event  $\{X(0) \in S_1\}$ , the process  $X$  up to time  $\tilde{\tau}$  is similar to  $X$  up to time  $\tau$ . In particular, given the event  $\{X(0) \in S_1\}$ ,  $X(0)$  is uniformly distributed over  $S_1$ . Let  $\hat{X}(t) = 2X(t/4)$ . Then given  $\{X(0) \in S_1\}$ ,  $\hat{X}$  is a Brownian motion on  $S$  with drift field  $\hat{b}(z) = \frac{1}{2}b(\frac{z}{2})$ , initial state uniformly distributed over  $S$ , and exit time from  $S$  equal to  $4\tilde{\tau}$ . Also,  $\|\hat{b}\|_1 = 2\Gamma_1$ . Therefore, if  $i = 1$

$$E[\tau | X(0) \in S_i] \geq \frac{1}{4}U(2\Gamma_i). \quad (46)$$

Similarly, (46) holds for  $2 \leq i \leq 4$ . Use  $P[X(0) \in S_i] = \frac{1}{4}$  for each  $i$ , to get

$$E[\tau] \geq \frac{1}{4} \sum_{i=1}^4 E[\tau | X(0) \in S_i] \quad (47)$$

and then use (46) and  $U(\Gamma) \geq E[\tau] - \epsilon$  to get

$$U(\Gamma) \geq \frac{1}{16} \sum_{i=1}^4 U(2\Gamma_i) - \epsilon. \quad (48)$$

Of course,

$$\Gamma = \sum_{i=1}^4 \Gamma_i \quad (49)$$

In summary,  $U(\Gamma)$  is a nonincreasing function of  $\Gamma \geq 0$  such that given any  $\Gamma, \epsilon > 0$ , there exist  $\Gamma_i \geq 0$  so that (48) and (49) hold. These facts about  $U$  are enough to complete the proof of the second half of Proposition 1.4.

Suppose that there exists  $v > 0$  and  $w > 0$  so that  $U(2w) = v$ . By monotonicity,  $U(\Gamma) \geq v$  for  $0 \leq \Gamma \leq 2w$ .

Given  $\Gamma, \epsilon \geq 0$ , let  $\lambda$  denote the probability measure on  $[0, \infty)$  with density  $\frac{1}{4} \sum_{i=1}^4 \delta_{2\Gamma_i}$ . That is,  $\lambda$  has point mass  $\frac{1}{4}$  at  $2\Gamma_i$  for each  $i$ . Equation (48) can be rewritten as

$$U(\Gamma) \geq \frac{1}{4} \int_0^\infty U(x) \lambda(dx) - \epsilon, \quad (50)$$

and (49) shows that  $\lambda \in M(\Gamma)$ , where  $M(\Gamma)$  is the set of all probability measures on  $[0, \infty)$  with mean  $\frac{\Gamma}{2}$ .

Let  $\mathcal{F}$  denote the class of nonnegative nonincreasing functions  $F$  on  $[0, \infty)$  such that  $F(x) = v$  for  $0 \leq x \leq 2w$ . Define the mapping  $\mathcal{T}$  from  $\mathcal{F}$  into  $\mathcal{F}$  as follows. For any  $F \in \mathcal{F}$ ,

$$\mathcal{T}F(\Gamma) = \begin{cases} v & \text{if } \Gamma \leq 2w \\ \inf\{\frac{1}{4} \int_0^\infty F(x) \lambda(dx) : \lambda \in M(\Gamma)\} & \text{if } \Gamma > 2w \end{cases} \quad (51)$$

The existence for each  $\Gamma > 2w$  and  $\epsilon > 0$  of a probability measure  $\lambda \in M(\Gamma)$  satisfying (50) implies that  $U \geq \mathcal{T}U$  pointwise. Note that the mapping  $\mathcal{T}$  is monotone, in that if  $F \leq G$  pointwise, then  $\mathcal{T}F \leq \mathcal{T}G$  pointwise. Let  $H_0$  denote the function in  $\mathcal{F}$  defined by  $H_0 = vI_{[0, 2w]}$ . By assumption,  $U \geq H_0$  pointwise. Let  $H_i$  denote the result of applying  $\mathcal{T}$  to  $H_0$   $i$  times. By induction on  $i$ ,  $U \geq H_{i+1} \geq H_i$  pointwise. Therefore the pointwise limit,  $H_\infty = \lim_{i \rightarrow \infty} H_i$  exists. The limit is a fixed point of  $\mathcal{T}$ :  $H_\infty = \mathcal{T}H_\infty$ , and  $U \geq H_\infty$ . It thus suffices to show that for some  $c > 0$ ,  $H_\infty(\Gamma) > \frac{c}{\Gamma^2}$  for all  $\Gamma > 0$ .

If  $\lambda \in M(\Gamma)$  then by Markov's inequality,  $\lambda([0, 3\Gamma/4]) \geq 1/3$ . Therefore,  $H_\infty(\Gamma) \geq \frac{1}{12} H_\infty(2\Gamma/3)$ . Consequently,  $H_\infty(\Gamma) > 0$  for all  $\Gamma$ .

It is shown next that  $\mathcal{T}F$  is convex over  $(2w, \infty)$  for any  $F \in \mathcal{F}$ . Given  $\Gamma = \theta\Gamma_1 + (1-\theta)\Gamma_2$  where  $0 < \theta < 1$  and  $2w < \Gamma_1 < \Gamma_2$ , and given  $\epsilon > 0$ , there are measures  $\lambda_i \in M(\Gamma_i)$  so that  $\mathcal{T}F(\Gamma_i) + \epsilon \geq \int_0^\infty F(x) \lambda_i(dx)$ . But the measure  $\lambda = \theta\lambda_1 + (1-\theta)\lambda_2$  satisfies  $\lambda \in M(\Gamma)$ . Therefore,

$$\begin{aligned} \mathcal{T}F(\Gamma) &\leq \int_0^\infty F(x) \lambda(dx) \\ &= \theta \int_0^\infty F(x) \lambda_1(dx) + (1-\theta) \int_0^\infty F(x) \lambda_2(dx) \\ &\leq \theta \mathcal{T}F(\Gamma_1) + (1-\theta) \mathcal{T}F(\Gamma_2) + \epsilon \end{aligned}$$

Since  $\epsilon$  can be arbitrarily small, it follows that  $\mathcal{T}F$  is convex over  $[2w, \infty)$  as claimed. The fixed-point relation implies therefore that  $H_\infty$  is convex over the interval  $(2w, \infty)$ .

The next step is to use Jensen's inequality to exploit the equality  $H_\infty = \mathcal{T}H_\infty$ . A slight problem occurs in that  $H_\infty$  is only convex on the interval  $(2w, \infty)$ , whereas the probability measures used in the definition of  $\mathcal{T}$  can have nonzero mass on the interval  $[0, 2w]$ . The following lemma addresses this problem, showing that if  $\Gamma \geq 6w$ , then only measures supported on  $(2w, \infty)$  are needed in computing  $\mathcal{T}H_\infty(\Gamma)$ . Define  $M'(\Gamma) = \{\lambda \in M(\Gamma) : \lambda((w, \infty)) = 1\}$ .

**Lemma 5.1** *If  $\Gamma \geq 6w$  then*

$$H_\infty(\Gamma) = \inf\left\{\frac{1}{4}\int_0^\infty H_\infty(x)\lambda(dx) : \lambda \in M'(\Gamma)\right\} \quad (52)$$

**Proof.** The measure consisting of a unit mass at  $\Gamma/2$  is in  $M(\Gamma)$  for any  $\Gamma > 0$ . Therefore,  $H_\infty(\Gamma) \leq \frac{1}{4}H_\infty(\Gamma/2)$  for  $\Gamma \geq 2w$ . In particular,  $H_\infty(3w) \leq \frac{v}{4}$ .

For the remainder of the proof of the lemma, let  $\Gamma \geq 6w$  and let  $\lambda \in M(\Gamma) - M'(\Gamma)$ . To prove the lemma it suffices to produce  $\lambda' \in M'(\Gamma)$  so that

$$\int_0^\infty H_\infty(x)\lambda(dx) - \int_0^\infty H_\infty(x)\lambda'(dx) \geq 0 \quad (53)$$

The idea is to shift all the probability mass of  $\lambda$  on the interval  $[0, 2w]$  and some of the probability mass of  $\lambda$  on the interval  $(\frac{\Gamma}{2}, \infty)$  (in a proportional way) to the point  $3w$ , to obtain  $\lambda'$ . The amount of mass to be transferred from  $(\frac{\Gamma}{2}, \infty)$  is determined by the requirement that  $\lambda'$  have mean  $\Gamma/2$ .

Specifically, for Borel measurable subsets  $E$  of  $[0, \infty)$ ,

$$\lambda'(E) = aI_{\{3w \in E\}} + \lambda(E \cap (2w, \frac{\Gamma}{2}]) + b\lambda(E \cap (\frac{\Gamma}{2}, \infty)) \quad (54)$$

where  $a$  and  $b$  satisfy

$$a = (1 - b)\lambda((\frac{\Gamma}{2}, \infty)) + \lambda([0, 2w]) \quad (\text{Conservation of probability}) \quad (55)$$

$$a(3w) = (1 - b)\int_{\frac{\Gamma}{2}+}^\infty x\lambda(dx) + \int_0^{2w+} x\lambda(dx) \quad (\text{Conservation of mean}) \quad (56)$$

Define  $\rho$  as the conditional mean of  $\lambda$  for the interval  $(\frac{\Gamma}{2}, \infty)$ . That is,

$$\rho = \frac{\int_{\frac{\Gamma}{2}+}^\infty x\lambda(dx)}{\lambda((\frac{\Gamma}{2}, \infty))}. \quad (57)$$

Then  $a$  and  $b$  can be expressed as

$$a = \frac{\rho\lambda([0, 2w]) - \int_0^{2w+} x\lambda(dx)}{\rho - 3w} \quad (58)$$

$$1 - b = \frac{3w\lambda([0, 2w]) - \int_0^{2w+} x\lambda(dx)}{(\rho - 3w)\lambda((\frac{\Gamma}{2}, \infty))} \quad (59)$$

It remains to check that  $b \geq 0$  (so that  $\lambda'$  is indeed a probability distribution), and that (53) is true.

The condition  $b \geq 0$  is equivalent to

$$3w \leq \frac{\int_0^{2w+} x\lambda(dx) + \int_{\frac{\Gamma}{2}+}^{\infty} x\lambda(dx)}{\lambda([0, 2w]) + \lambda((\frac{\Gamma}{2}, \infty))}. \quad (60)$$

The left side of (60) expresses the conditional mean of  $\lambda$  for the set  $[0, \infty) - (2w, \frac{\Gamma}{2})$ . Since the omitted interval  $(2w, \frac{\Gamma}{2})$  lies to the right of the mean,  $\frac{\Gamma}{2}$ , of  $\lambda$ , the left side of (60) is greater than or equal to  $\frac{\Gamma}{2}$ , whereas  $3w \leq \frac{\Gamma}{2}$ . Thus,  $b \geq 0$  is proved.

Finally, the facts  $H_{\infty}(3v) \leq \frac{v}{4}$ , and  $3w < \frac{\Gamma}{2} < \rho$  and some algebraic manipulation yield

$$\begin{aligned} \text{Left side of (53)} &= v\lambda([0, 2w]) + (1 - b) \int_{\frac{\Gamma}{2}+}^{\infty} U_{\infty}(x)\lambda(dx) - aH_{\infty}(3v) \\ &\geq v\lambda([0, 2w]) - a\frac{v}{4} \\ &= \frac{v}{\rho - 3w} \left[ \left(\frac{\rho}{\Gamma} - 1\right)3w\lambda([0, 2w]) + \left(1 - \frac{3w}{\Gamma}\right) \int_0^{2w+} x\lambda(dx) \right] \geq 0. \end{aligned}$$

The proof of the lemma is complete. ■

The proof of Proposition 1.4 can now be completed. The convexity of  $H_{\infty}$  over  $(2w, \infty)$  and Jensen's inequality imply that for  $\Gamma \geq 6w$  and  $\lambda \in M'(\Gamma)$ ,  $\int_0^{\infty} H_{\infty}(x)\lambda(dx) \geq H_{\infty}(\Gamma/2)$ . Therefore, by Lemma 5.1

$$H_{\infty}(\Gamma) \geq \frac{1}{4}H_{\infty}\left(\frac{\Gamma}{2}\right) \quad \text{for } \Gamma \geq 6w. \quad (61)$$

So for any integer  $n \geq 0$ ,  $H_{\infty}(\Gamma) \geq \frac{1}{4^{n+1}}H_{\infty}(6w)$  for  $\Gamma \in ((6w)2^n, (6w)2^{n+1}]$ . Thus,  $H_{\infty}(\Gamma) \geq \frac{(6w)^2 H_{\infty}(6w)}{4\Gamma^2}$  for all  $\Gamma \geq 6w$ . The proof of Proposition 1.4 is complete.

## 6 Discussion

Together Propositions 1.2 and 1.3 show that for large  $\Gamma$ , angularly symmetric drift fields are far from optimal for the problem of hitting the center of a disk. Likewise, Proposition 1.4 shows that for large  $\Gamma$ , constant drift is far from optimal for the problem of exiting a square.

It would be nice to settle Conjecture 1.1. Propositions 1.3 and 1.4 combined provide strong evidence for the conjecture. By the second part of Proposition 1.4, if Conjecture 1.1 is false, then there must exist drift fields that are much, much better than the bicycle wheel vector fields used to establish Proposition 1.3. On the other hand, it is difficult to imagine a structure that can improve on the bicycle wheel construction by any more than a constant factor. Two ideas for improving on the bicycle wheel structure are commented on in the next two paragraphs. Both ideas aim to reduce  $E[\tau_1]$ , the mean time to first reach the center of a spoke, since by far it is the dominant term in  $E[\tau_{\{0\}}]$ .

One idea is to use more spokes. For example, consider increasing the the number of spokes  $M$  to be proportional to  $\Gamma$ . Then the integral across a spoke of the magnitude of the component of drift pointing into a spoke, namely  $2Lf$ , would be bounded, rather than growing as  $\log \Gamma$ . However, the growth rate  $\log \Gamma$  is needed to hold the particle on the spokes for a period of time on the order of  $\frac{1}{\Gamma^2}$ , because the large deviations bound of Lemma 2.1 is sufficiently tight. Thus, this idea doesn't seem to work.

A second idea is to replace the spokes by a tree of highways. This would avoid the close spacing of spokes near the center of the circle. However, the total length of highway would be limited to  $\frac{\Gamma}{\log \Gamma}$  by the considerations of the previous paragraph, so that still most points of the disk would be distance at least  $\frac{\Gamma}{\log \Gamma}$  from the nearest highway. One way to think about this is that the spokes are rather well spaced outside the disk of radius 0.5 about the origin, and only half the total length of spoke is used on the inner disk. Thus, the second idea also does not lead to more than a constant factor reduction in the mean hitting time.

A brief comment on compactification is given next. Section 2 shows that a natural compactification exists for the problems examined in the one-dimensional case. Specifically, the set of drift fields considered are the set of vector valued measures with total variation at most  $\Gamma$ . It is natural to consider such vector fields for the two-dimensional problems as well. While significant progress has been made recently on the construction of Brownian motion with singular drift (see [1]

and references therein), the theory does not cover all bounded variation vector valued measures for the drift field. An alternative approach, which might yield interesting results, would be to consider other ways to constrain the drift field, for example constraining the  $L^2$  norm, or simultaneously constraining the  $L^1$  and  $L^\infty$  norms.

Finally, a comment is given on how the diffusion model studied in this paper relates to a discrete time random walk model. Suppose  $v$  is a unit vector of dimension two, and suppose  $p > 0$ . Consider a random walk  $X(k)$  on the plane with initial state 0 such that the step  $X(k+1) - X(k)$  takes the value  $v$  with probability  $p$ , and otherwise is normally distributed with mean zero and covariance matrix  $I$ , the  $2 \times 2$  identity matrix. The interpretation is that with probability  $p$  the particle “knows” to take the unit step  $v$ , and otherwise it takes a mean zero, normally distributed step. Let  $\kappa > 0$ . Given  $0 < \epsilon \leq 1/\kappa$ , take  $p = \epsilon\kappa$  and let  $X^\epsilon(t) = \epsilon X(\lfloor \frac{t}{\epsilon^2} \rfloor)$ . Then by the functional central limit theorem, as  $\epsilon \rightarrow 0$ ,  $X^\epsilon$  converges weakly to a Brownian motion process with constant drift  $\kappa v$ . Note that the magnitude of the drift,  $\kappa$ , is proportional to the probability that the particle “knows” to take the jump  $v$ . This motivates our choice of using the  $L^1$  norm in the hitting time problem formulated in Section 1. Also note that, in the limiting regime, the probability  $p$  that the particle “knows” to go in direction  $v$  at a particular time tends to zero for any fixed  $\kappa$ .

The mean and covariance matrix of  $X^\epsilon(1)$ , assuming  $1/\epsilon$  is an integer, are  $\kappa v$  and  $\Sigma_\epsilon = vv^T \epsilon\kappa(1 - \epsilon\kappa) + (1 - \epsilon\kappa)I$ , respectively. Note that for fixed  $\epsilon$ , if  $\kappa$  approaches  $1/\epsilon$ , the covariance matrix tends to zero. This reflects the fact that, for the random walk model, the more knowledge the particle has about what direction to go (so the larger the drift), the smaller the covariance matrix. This may lead one to consider diffusion processes such that the local diffusion matrix is a function of the drift. In contrast, the diffusion model in this paper, as noted in the previous paragraph, corresponds to the case that only a small fraction of nodes visited have a preferred direction for particle movement.

## 7 Appendix: Mathematical Underpinnings of Brownian Motion with Drift on a Disk

Proposition 1.1 is proved in this section. The proposition falls within the well known theory of constructing diffusions by stochastic differential equations [2, 4, 5, 8]. A diffusion on a halfspace with more general boundary conditions is constructed in [4], and the case of a disk is similar, so



only the main points are given here.

The proposition is proved first under the assumption that the drift field  $b$  is identically zero. (For this purpose we could use Theorem 8.1.5 of Ethier and Kurtz [2], which identifies a somewhat different generator of a Feller semigroup for a nondegenerate diffusion on a domain with smooth boundary. Instead, we take a more direct approach.) The questions of existence and uniqueness are local questions [8, Section 6.6], so they can be settled by considering the process  $X$  until it leaves a neighborhood of  $x$ . Existence and uniqueness are evident if  $\|x\| < R$  and the process  $X$  is considered until the first time that  $\|X(t)\| = R$ , because  $X$  coincides with  $B(t)$  until that time. So suppose that  $\|x\| > R/2$ . The process  $X$  will be considered up until the first time  $\tau$  that  $\|X(t)\| = R/2$ .

Suppose there exists a solution for  $x$ , and let  $(R(t), \Theta(t))$  denote the solution in polar coordinates. Ito's differentiation formula can be applied to obtain the semimartingale representation for  $R(t), \Theta(t)$  as follows:

$$\begin{pmatrix} dR(t) \\ d\Theta(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{R(t)} \end{pmatrix} d\hat{B}(t) + \begin{pmatrix} \frac{1}{2R(t)} \\ 0 \end{pmatrix} dt - \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\phi(t) \quad (62)$$

where  $\hat{B}$  is the new Brownian motion defined by

$$d\hat{B}(t) = \frac{1}{R(t)} \begin{pmatrix} X_1(t) & X_2(t) \\ -X_2(t) & X_1(t) \end{pmatrix} dB(t) \quad (63)$$

The next step is to use the Girsanov change of measures method [4, Section IV.4] to eliminate the drift term in (62). Define the martingale  $\mu$  by

$$\mu(t) = \exp\left(-\int_0^{t \wedge \tau} \frac{1}{2R(s)} d\hat{B}_1(t) - \frac{1}{2} \int_0^{t \wedge \tau} \frac{1}{(2R(s))^2} ds\right) \quad (64)$$

and define a process  $\tilde{B}$  by

$$\tilde{B}(t) = \hat{B}(t) + \int_0^{t \wedge \tau} \begin{pmatrix} \frac{1}{2R(s)} \\ 0 \end{pmatrix} ds, \quad (65)$$

Then  $\tilde{B}$  is a Brownian motion under the new probability measure  $\tilde{P}$  such that  $\frac{d\tilde{P}}{dP}$  for  $(\Omega, \mathcal{F}_t)$  is

given by  $\mu(t)$ . Equation (62) for  $t \leq \tau$  becomes simply

$$\begin{pmatrix} dR(t) \\ d\Theta(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{R(t)} \end{pmatrix} d\tilde{B}(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\phi(t) \quad (66)$$

where  $\phi$  is nondecreasing and

$$\int_0^t I_{\{R(s)=R\}} d\phi(s) = \phi(t) \quad (67)$$

Equations (66) and (67) are rather simple. The first coordinate of (66) together with (67) are the Skorohod mapping equations [4, Theorem III.4.2], showing that under  $\tilde{P}$ ,  $R - R(t)$  is reflecting Brownian motion (up until it hits  $R/2$ ). The second coordinate equation of (66) expresses the process  $\Theta$  simply as a stochastic integral with  $1/R(t)$  integrated against a Brownian motion independent of  $R(t)$ . Thus, the distribution of  $(R(t), \Theta(t) : 0 \leq t \leq \tau)$  under  $\tilde{P}$  is uniquely specified. Conversely, given processes on some probability space  $(\Omega, \mathcal{F}, \tilde{P})$  with filtration  $(\mathcal{F}_t : t \geq 0)$ , with this distribution of  $(R(t), \Theta(t) : 0 \leq t \leq \tau)$ , the steps taken can be reversed to yield a solution for the original equations. Thus, the law of  $(X(t) : 0 \leq t \leq \tau)$  under  $P_x$  is uniquely specified if  $\|x\| \geq R/2$ . Hence  $P_x$  is uniquely determined for all initial conditions  $x \in D$ , under the assumption that  $b$  is identically zero.

Next it is shown how to drop the added assumption that  $b(x)$  is zero. The idea is to again use the Girsanov change of measures method. Solutions of the stochastic differential equation with bounded measurable  $b$  can be constructed from solutions for  $b$  identically zero, and vice versa, by a change of measure. Therefore existence and uniqueness in law for bounded measurable  $b$  follows from the existence and uniqueness in law already established. See [4, Section IV.4] for details.

It remains to establish the strong Markov property. For that purpose consider the following martingale problem (see [4, Definition IV.7.1]). A system  $\{P_x : x \in D\}$  of probabilities on  $\mathcal{B}(C([0, \infty), R^2))$  satisfies the martingale problem for Brownian motion in  $D$  with drift  $b$  if

- (i)  $P_x[w : w(0) = x] = 1$ ;
- (ii) there exists an adapted function  $\phi(t, w)$  defined on  $[0, \infty) \times \mathcal{B}(C([0, \infty), R^2))$  such that
  - (a) with  $P_x$  probability one,  $\phi(0, w) = 0$  and  $\phi(w, t)$  is nondecreasing in  $t$ , and

$$\int_0^t I_{\{X(s)=R\}} d\phi(s, w) = \phi(t, w) \quad \text{for all } t \geq 0, \quad (68)$$

(b) For any twice continuously differentiable function  $f$  on  $D$ , with inward normal derivative  $\frac{\partial f}{\partial \eta}$  on  $\partial D$ ,

$$f(w(t)) - f(w(0)) - \int_0^t \left( \frac{1}{2} \Delta f \right)(w(s)) + b(w(s))^T \nabla f(w(s)) ds - \int_0^t \frac{\partial f}{\partial \eta}(w(s)) d\phi(s, w) \quad (69)$$

is a  $P_x$  martingale.

By use of Ito's formula and a characterization of Brownian motion, it can be shown that the question of existence and uniqueness for the martingale problem is equivalent [4, Theorem IV.6.1] to the stochastic differential equation problem, already settled above. The strong Markov property for the family  $\{P_x\}$  then follows [4, Theorem 4.5.1] from the existence and uniqueness of the solutions for a martingale problem.

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