Second, we evaluate the third term in the RHS of (A.1). Choose an arbitrary constant $\epsilon > 0$. Then, define the sets $\mathcal{P}_\epsilon$, $\mathcal{N}_\epsilon$, and $\mathcal{S}_\epsilon$ as follows: $\mathcal{P}_\epsilon \triangleq \{t \in [0,T] | \dot{x}^d(t) \geq \epsilon\}$, $\mathcal{N}_\epsilon \triangleq \{t \in [0,T] | \dot{x}^d(t) \leq -\epsilon\}$, and $\mathcal{S}_\epsilon \triangleq \{t \in [0,T] |\, |\dot{x}^d(t)| < \epsilon\}$. By Proposition 3.24 in [1] along with (15), a subset $\mathcal{E}$ of $[0,T]$ and an integer N exist such that

$$\mu(\mathcal{E}) < \epsilon, \quad |\dot{\overline{x}}_{i-1} - \dot{x}^d| < \epsilon, \quad \forall t \in [0,T] - \mathcal{E}, \quad \forall i \geq N. \tag{A.3}$$

Furthermore, it follows from (A.3) and the definition of $\mathcal{B}_i$ that, if $i \geq N$, $t \in \mathcal{B}_i$ implies that $t \notin (\mathcal{P}_\epsilon - \mathcal{E}) \cup (\mathcal{N}_\epsilon - \mathcal{E})$ and, hence, that $t \in \mathcal{S}_\epsilon \cup \mathcal{E}$. Therefore, we conclude that

$$\mathcal{B}_i \subset \mathcal{S}_\epsilon \cup \mathcal{E}, \qquad \forall\, i \geq N. \tag{A.4}$$

On the other hand, we see from (13) that $\lim_{\epsilon \to 0} \mu(\mathcal{S}_\epsilon) = 0$. By (A.3) and (A.4), this result implies that $\lim_{i \to \infty} \mu(\mathcal{B}_i) = 0$. Therefore, we have

$$\lim_{i \to \infty} \int_{\mathcal{B}_i} (\overline{F}_i(\tau) - \overline{F}_{i-1}(\tau))\, d\tau = 0. \tag{A.5}$$

Finally, we evaluate the last term in the RHS of (A.1). The above function $\overline{F}_i - \overline{F}_{i-1}$ is continuous at each $t \in \mathcal{C}_i$. Therefore, for any $\epsilon > 0$, an integer N exists such that $|\overline{F}_i(t) - \overline{F}_{i-1}(t)| < \epsilon, \forall t \in \mathcal{C}_i$, $\forall\, i \geq N$. This result implies that

$$\lim_{i \to \infty} \int_{\mathcal{C}_i} (\overline{F}_i(\tau) - \overline{F}_{i-1}(\tau))\, d\tau = 0. \tag{A.6}$$

Now, (28) is the immediate consequence of (A.2), (A.5), and (A.6). ■

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Large Bursts Do Not Cause Instability

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***Abstract*—It is shown that stability of networks with fluid traffic implies stability of networks with deterministically constrained traffic.**

***Index Terms*—Fluid traffic, queueing network, stability.**

I. INTRODUCTION

Fluid models of queuing networks are among the simplest models to analyze, owing to the fact that calculus can be applied. At the same time, wider classes of network models are more flexible for modeling real traffic. It is thus useful to reduce questions about the more realistic models to questions about related fluid models. Such a reduction was achieved by Dai [4], who showed that stability of a fluid model implies stability (in the sense of Harris recurrence) of related multiclass networks with random service and interarrival processes of renewal type. The purpose of this correspondence is to similarly reduce the question of stability for networks with input traffic satisfying *deterministic constraints* in the sense of Cruz [2] to a question of stability for a fluid model.

II. THE NETWORK MODEL

The network we consider has d single server stations and K classes of traffic. Class l traffic is served at a unique station $s(l)$. Let C be the $d \times K$ matrix such that $C_{i,l} = 1$ if $s(l) = i$ and $C_{i,l} = 0$ otherwise. Upon completion of service at $s(l)$, the traffic of class l either becomes traffic of class l' for some other class l', in which case, we write $l \to l'$, or it immediately exits the network. Let P denote the $K \times K$ matrix such that $p_{l,l'} = 1$ if $l \to l'$ and $p_{l,l'} = 0$ otherwise. A simple network with three stations and eight classes is shown in Fig. 1 with $1 \to 2 \to 3$, $4 \to 5 \to 6$, and $7 \to 8$ and $s = (1,2,3,2,3,1,3,2)$. It is assumed that the network is open, so that P^K is the zero matrix.

Exogenous traffic can enter the network as any class, though for the example given it might make sense for the exogenous arrival functions to be nonzero only for classes 1, 4, and 7. Let $E_l(t)$ denote the amount of exogenous class l traffic to enter the network during $[0,t]$. Suppose

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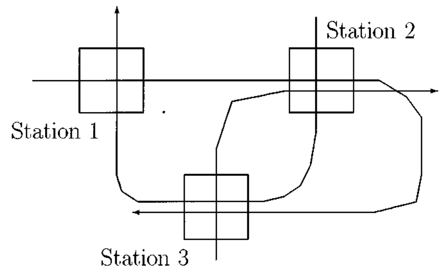

Fig. 1. Sample network.

that $E_l(0) = 0$ and that E_l is nondecreasing and right-continuous. Traffic of class l can be served [at station $s(k)$] at a maximum rate $\mu_l = 1/m_l$, where $m_l > 0$. Let $M = \text{diag}\,(m_1, \cdots, m_K)$, and let e denote a column vector of all ones (with dimension depending on the context). The flow of traffic in the network is assumed to satisfy the following equations and conditions:

$$Q(t) = q + E(t) + (P^T - I)M^{-1}T(t) \tag{1}$$

$$I(t) = et - CT(t) \tag{2}$$

$$Q(t) \geq 0, \quad \text{for } t \geq 0 \tag{3}$$

$$\int_0^\infty (CQ(t) \wedge e)\, dI(t) = 0 \tag{4}$$

$$T(0) = 0, \quad T \text{ is right-continuous and nondecreasing} \tag{5}$$

$$I(0) = 0, \quad I \text{ is right-continuous and nondecreasing.} \tag{6}$$

The following interpretations hold.

- $Q_l(t)$ is the amount of class l traffic in the network at time t, and $Q_l(0) = q_0$.
- $T_l(t)$ is the amount of work (where work is measured in units of time) done on class l traffic during $[0, t]$. A unit increment of T_l corresponds to μ_l units of class l traffic served at station $s(l)$, which becomes class l' traffic if $l \to l'$.
- $(CT(t))_i$, where $CT(t)$ is simply the matrix C times the vector $T(t)$, is the amount of work (where work is measured in units of time) done at station i during $[0, t]$.
- $I_i(t)$ is the amount of idleness (measured in units of time) of the server at station i accumulated during $[0, t]$.
- $(CQ(t))_i$ is the amount of traffic at station i at time t.

The exogenous traffic E is said to satisfy deterministic constraints with parameters $\alpha = (\alpha_l)$ and $\sigma = (\sigma_l)$, abbreviated to "E is DC(α, σ) traffic," if

$$0 \leq E_l(t) - E_l(s) \leq \alpha_l(t-s) + \sigma_l \qquad 0 \leq t \leq s < \infty. \tag{7}$$

This work was completed independently of the work of Gamarnik [7], which is remarkably similar. Some differences exist, aside from differences in terminology. The arrival constraints imposed by Gamarik are more general, in that they are imposed on sums of path arrival processes, one constraint for each link in the network, whereas we assume each individual arrival stream is constrained. On the other hand, the type of constraint we impose is somewhat more general than that of Gamarik, in that the Cruz type contraints used here are more general than constraining the number of arrivals in fixed length windows. Also, the service times for packets on a given path can be different at different stations.

III. Sufficient Conditions for Stability

Definition 1: The network (C, P, m) is totally stable for DC(α, σ) traffic if a finite constant Γ exists so that whenever (E, q, Q, T, I) satisfy (1)–(7), then $\lim\ \sup_{t\to\infty} |Q(t)| \leq \Gamma$, where $|Q(t)| = \sum_l |Q_l(t)|$. We use the word "totally" in the above definition because the constraints do not indicate an order of service at the stations.

A subclass of DC(α, σ) traffic is DC$(\alpha, 0)$ traffic, obtained by setting the vector $\sigma = 0$. DC$(\alpha, 0)$ traffic can be considered to be fluid traffic with arrival rates *bounded* by α. The special case $E(t) = t\alpha$ is the fluid arrival function with arrival rate vector *equal* to α. For fluid arrivals, we adopt a different definition of stability, though it is not difficult to show that it is equivalent to the one above.

Definition 2: The network (C, P, m) is totally stable for traffic with arrival rates bounded by α (respectively, exactly equal to α) if a finite constant τ_f exists so that whenever (E, q, Q, T, I) satisfies (1)–(6), $|q| \leq 1$, and E is DC$(\alpha, 0)$ traffic (respectively, $E(t) = \alpha t$), then $Q(t) = 0$ for $t \geq t_f$.

Two propositions are given below. The first reduces the question of stability for DC(α, σ) traffic to a question of stability for fluid traffic with arrival rate vector bounded by α, and the second proposition further reduces it to a question of the stability for fluid traffic with arrival rate vector equal to α. The proofs are found in the Appendix. Proposition 2, basically a "monotonicity" result, is a special case of Proposition 3.6 in [3]. Not all details are provided in the proof here.

Proposition 1: If the network (C, P, m) is totally stable for $DC(\alpha, 0)$ traffic, then it is totally stable for $DC(\alpha, \sigma)$ traffic, for any vector σ.

Proposition 2: If the network (C, P, m) is totally stable for fluid traffic of rate α, then it is totally stable for $DC(\alpha, 0)$ traffic.

IV. Conclusions

This work was originally motivated by the juxtaposition of two interesting results: an ingenious but complex proof of Tassiulas and Georgiades [8] of total stability for deterministically constrained traffic on a unidirectional ring network, on the one hand, and an ingenious and simple proof later obtained by Dai and Weiss [5] that the same network is stable for fluid traffic, on the other hand. A unidirectional ring network is characterized by the constraint that $l \to l'$ only if $l' = l + 1$ mod K. Propositions 1 and 2 combined with the stability of ring networks with fluid traffic whenever $\rho = M(I - P)^{-1}\alpha < e$ yields another proof of part of the result in [8]. The result in [8] is stronger, however, in that it provides a simple explicit bound on the lim sup of the maximum queue length. (It also allows for nonzero transmission delays, but it assumes that processor speeds are all one.) The proof of stability for fluid traffic in [5] for the ring network offers a clean bound on the rate of evacuation. It would be desirable if Proposition 1 could be strengthened to allow us to translate bounds for evacuation rates in fluid models to bounds on asymptotic queue lengths with deterministically constrained traffic.

Propositions 1 and 2 can apparently be generalized to cover networks with specified classes of service orders, which are represented by additional constraints. For example, it is known that fluid networks with FIFO service order are stable if $\rho < e$ and $m_k = m_l$ whenever $s(k) = s(l)$ [1]. If this conjecture is proved, an extension of Proposition 1 could show that such networks are stable for deterministically constrained traffic as well. In order to extend Proposition 1 to hold when the networks are restricted to a class of networks using a specified service order, the constraint on order of service should be inherited under u.o.c. convergence of (E, q, Q, T, I) and other functions that may be needed to model the network with a specific order of service constraints.

Appendix

The two propositions are proved in this Appendix. A key to the proof of Proposition 1 is that, if E is DC(α, σ) traffic, then for any $\gamma > 0$ the rescaled function $t \to E(\gamma t)/\gamma$ is DC$(\alpha, \sigma/\gamma)$ traffic. Moreover, the conditions (1)–(6) are invariant under a similar rescaling of (E, Q, T, I). The proposition is proved after the following lemma is proved.

Lemma 1: Let $f \in D_{R+}[0, \infty)$ such that for some nonnegative constants $\tau, \delta, a, b, c,$ with $\delta < 1$

$$f(t + \tau f(t)) \le \delta f(t), \quad \text{if } f(t) > a \tag{8}$$

$$f(t + s) - f(s) \le b + ct, \quad \text{if } s, t \ge 0. \tag{9}$$

Then, $f(s) \le (a + b)\,(1 + c\tau/(1 - \delta))$ for $s \ge f(0)\,(\tau/(1 - \delta))$.

Proof: If $f(t) > a$, then by (8)), $f(t + \tau f(t)) \le f(t) - (1 - \delta/\tau)\,(\tau f(t))$. Graphically, if $f(t) > a$, then $(t+\tau f(t), f(t+\tau f(t)))$ lies below the line through $(t, f(t))$ with slope $-(1-\delta)/\tau$. Therefore, defining the sequence of points $(t_n, f(t_n))$ by $t_0 = t$ and $t_{n+1} = t_n + \tau f(t_n)$, and letting $N = \min\{n \ge 0\colon f(t_n) \le a\}$, the points of the sequence $t_0, \cdots, t_N$ all lie below the line through $(t, f(t))$ with slope $-(1-\delta)/\tau$. Therefore, $f(t) - (t_N - t)\,((1-\delta)/\tau) \ge f(t_N) \ge 0$ so that $t_N \le t + f(t)\,(\tau/(1-\delta))$. Thus, if $f(t) > a$, then $f(s) \le a$ for some $s \in [t, t+f(t)\,(\tau/(1-\delta))]$. In particular, the time t^* defined by $t^* = \min\{s \ge 0\colon f(s) \le a\}$ is at most $f(0)\,(\tau/(1-\delta))$. To finish the proof, a bound on f restricted to the interval $[t^*, \infty)$ needs to be found. Let $a' > a$. If $f(t-) \le a' \le f(t)$, then by condition (9), $f(t) \le f(t - \epsilon) + b + c\epsilon$ for any $\epsilon > 0$ so that $f(t) \le a' + b$. Thus, by the first paragraph of the proof, $f(s) \le a$ for some s in the interval $[t, t + (a' + b)\,(\tau/(1 - \delta))]$. By (9), throughout this interval, $f(s) \le a' + b + c(a' + b)\,(\tau/(1-\delta))$. The whole set $\{s \ge t^*\colon f(s) \ge a'\}$, however, is covered by such intervals. Conclude that for $s \ge t^*$, $f(s) \le (a' + b)\,(1 + c\tau/(1 - \delta))$. Because $a' > a$ is arbitrary, the proof of the lemma is complete.

Proof of the Proposition 1: Suppose (C, P, m) is not totally stable for DC(α, σ) traffic. Let $b = |\alpha + (I + P^T)\mu|$, $c = |\sigma|$, and fix $\tau > 0$ and δ with $0 < \delta < 1$. By Lemma 1 and the assumption that (C, P, m) is not totally stable for DC(α, σ) traffic, given $a > 0$ a solution $\mathcal{S}^a = (E^a, Q^a, q^a, T^a, I^a)$ to (1)–(7) exists such that f^a defined by $f^a(t) = |Q^a(t)|$ violates either (8) or (9). Examination of the constraints (1), (2), (5), and (6) satisfied by $\mathcal{S}^a$ shows that f^a satisfies (9), so it must violate (8), meaning $f^a(t_a + \tau f^a(t_a)) > \delta f^a(t_a)$ and $f(t_a) > a$ for some $t_a \ge 0$. By shifting the functions of $\mathcal{S}^a$ to the left if necessary, it can be assumed without loss of generality that $t_a = 0$ for all a. Hence, $\mathcal{S}^a = (E^a, Q^a, q^a, T^a, I^a)$ satisfies (1)–(7) and the additional constraints $|Q^a(\tau|q^a|)| > \delta|q^a|$ and $|q^a| \ge a$.

Next, a space-time rescaling is used. Let $\overline{E}^a(t) = E^a(t|q^a|)/|q^a|$, and define $\overline{Q}^a$, $\overline{T}^a$, and $\overline{I}^a$, similarly. Also, set $\overline{q}_a = q_a/|q_a|$. Note that $|\overline{Q}^a(0)| = |\overline{q}^a| = 1$ and $|Q^a(\tau)| \ge \delta$ for all a. The scaled versions $(\overline{E}^a, \overline{q}^a, \overline{Q}^a, \overline{T}^a, \overline{I}^a)$ again satisfy (1)–(6), and (7) is satisfied with σ replaced by $\sigma/|q_a|$, which is at most σ/a. The family of functions $(\overline{E}^a)_{a \ge 0}$ is thus asymptotically uniformly equicontinuous as $a \to \infty$, and $|\overline{E}^a(t)| \le |\alpha|t + |\sigma|$ for all $a \ge 1$, so the family is also asymptotically uniformly bounded on compact intervals. The functions $|\overline{T}^a|$ are Lipschitz continuous uniformly in a with common value zero at the origin. Thus, continuous limit functions exist such that, along a subsequence of $a \to \infty$, the following limits hold: $\overline{q}^a \to q$, $\overline{E}^a \Rightarrow E$ uniformly on compact intervals (u.o.c.) and $\overline{T}^a \Rightarrow T$ u.o.c. Note that $|q| = 1$. With Q and I defined in terms of E and T by (1) and (2), it also follows that $(\overline{Q}^a, \overline{I}^a) \to (Q, I)$ u.o.c. along the same subsequence. The limit (E, q, Q, T, I) satisfies the conditions (1)–(6). [To verify condition (4) use the fact that if $((z_n, y_n))$ is a sequence in $D_R[0, \infty) \times D_R[0, \infty)$, y_n in nondecreasing for each n, and $(z_n, y_n) \to (z, y)$ u.o.c. then for any bounded continuous function h, $\int_0^t h(z_n(s))\, dy_n(s) \Rightarrow \int_0^t f(z(s)\, dy(s)$ u.o.c. [6, Lemma 2.4].] The fact that $\overline{E}^a$ is DC$(\alpha, \sigma/a)$ traffic for each a yields in the limit that E is DC$(\alpha, 0)$ traffic. Finally, because $|\overline{Q}^a(\tau)| \ge \delta$ for all a, the limit satisfies $Q(\tau) \ge \delta$.

To summarize, it has been shown that if (C, P, m) is not totally stable for DC(α, σ) traffic, then for any $\tau > 0$ and any δ with $0 < \delta < 1$, a solution (E, q, Q, T, I) to (1)–(6) exists such that E is DC$(\alpha, 0)$ traffic, $|Q(0)| = |q| = 1$, and $|Q(\tau)| \ge \delta$. Therefore, (C, P, m) is not totally stable for DC$(\alpha, 0)$ traffic. The contrapositive of the proposition, and, hence, the proposition itself, is proved. □

Proof of Proposition 2: It suffices to show that given $\mathcal{S} = (E, q, Q, T, I)$ satisfying (1)–(6), where E is DC$(\alpha, 0)$, another solution $\tilde{\mathcal{S}} = (\tilde{E}, q, \tilde{Q}, \tilde{T}, \tilde{I})$, can be found with $\tilde{E}(t) = \alpha t$ such that $Q(t) \le \tilde{Q}(t)$ for all t. Intuitively, because $E(t) - E(s) \le \tilde{E}(t) - \tilde{E}(s)$ whenever $s < t$, the traffic for $\tilde{\mathcal{S}}$ can be viewed as the sum of two traffic types: original traffic and "ghost" traffic. The servers simply give priority to original traffic. The solution $\tilde{\mathcal{S}}$ is easy to construct by induction if the functions comprising $\mathcal{S}$ are piecewise linear, and in the general case, a sequence of approximating solutions is used. The details are similar to those in the proof of Proposition 1 and are omitted. □

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