A New Upper Bound to the Throughput of a Multi-Access Broadcast Channel

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Abstract—A new upper bound of 0.6126 packets/slot is established for the through-put of a time-slotted multi-access broadcast channel subject to an infinite population of user stations (whose transmissions are modeled by a Poisson process) using (0)-, (1)-, (e)-feedback to denote a slot with none, one, or at least two packets, respectively.

I. Introduction

THE GENERATION of information packets by remote stations will be modeled by a Poisson point process $\{a \leq \alpha_1 < \cdots < \alpha_W \leq b\}$ on the time interval, $T = [a, b]$, with intensity $\lambda \geq 0$. Each of the W packets can thus be identified with the time that it was generated. A conflict resolution algorithm (CRA) is a protocol that the remote stations follow to access a central broadcast channel. Time is divided into slots numbered $k = 1, 2, \cdots$. At the beginning of the kth slot, the CRA designates a subset φ_k of T. Each packet in φ_k is then transmitted during that slot. A transmitted packet successfully reaches its destination only if no other packet is transmitted in the same slot. If more than one packet is transmitted in a slot, the packets "collide" and must be retransmitted.

It is assumed that, by listening to the channel output during slot k, each station learns Z_k, where $Z_k = 0$, $Z_k = 1$, or $Z_k = e$, depending on whether zero, one, or more than one packet was transmitted during the slot. The set φ_k specified by the CRA must be a function of the past channel information $(Z_1,\cdots,Z_{k-1})$ so that the algorithm can be implemented in a distributed fashion. See [3] for other interpretations of the interval $[a, b]$ and for an explanation of why this framework includes randomized strategies such as the basic Capetanakis–Tsybakov–Mikhailov algorithm [2], [10], [1], [5], where the users make decisions on the basis of independent coin flips [5], as well as on the feedback information.

The algorithm is completed when it becomes known that all packets have been successfully transmitted. Let τ be the (random) number of slots until completion. The efficiency of a CRA is defined to be $\eta = E[W]/E[\tau]$. The purpose of this paper is to prove the following theorem.

Theorem: $\eta < 0.6126$ for any CRA.

Using information theoretic arguments, Pippenger [9], [1] obtained an upper bound of 0.744 on η, and his argument was sharpened by Hajek [3] to yield a bound of 0.711. Humblet [4] similarly established an upper bound of 0.704. Subsequently, Molle [6], [7] introduced a "genie" argument to establish an upper bound of 0.674. After the original submission of the present paper, an upper bound of 0.587 was established by Tsybakov and Mikhailov [11]. One part of their argument is similar to our argument following the lemma in Section III below. They also have a counterpart to our lemma, but their state is different from ours in that it includes the amount of the interval T which has been completely processed by slot k. We believe that by combining the lemmas of [11] and the present paper, a still lower bound can be obtained. The greatest known efficiency that can be achieved by CRA's for all large $E[W]$ is about 0.488, and this is known to be the maximum achievable asymptotic efficiency in the class of all first-come first-serve single-interval CRA's [8].

II. Auxiliary Information and States of Knowledge

One idea of our proof, suggested first by Molle [6], is to consider the CRA's which use certain additional information which is not provided under the original channel model. More specifically, at the end of slot k all stations are told to subset $\mathcal{Q}_k$ of the set $\{\alpha_1,\cdots,\alpha_W\}$ of packet locations, where $\mathcal{Q}_k$ is determined according to the rules specified below. We will then allow the set φ_k chosen by a CRA to depend on $Z_1,\cdots,Z_{k-1}$, $\mathcal{Q}_1,\cdots,\mathcal{Q}_{k-1}$. Since the auxiliary information could be ignored, any upper bound on the efficiency of such possibly unrealizable algorithms is also an upper bound on the efficiency of the original class of algorithms. Our bound is tighter than that of Molle [6], since we consider a smaller amount of auxiliary information.

Note that φ_k can be viewed as a random set. Throughout this paper, we consider only random sets that are countable unions of random intervals (which may be open, semi-open, or closed), whose endpoints consist of random variables. In this way, a random set may be identified with a countable collection of random variables. The condition that φ_k can "depend" on $Z_1,\cdots,Z_{k-1}$, $\mathcal{Q}_1,\cdots,\mathcal{Q}_{k-1}$ means precisely that φ_k (or rather, each of the random variables identified

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TABLE I
THE SET $\mathcal{Q}_k$ AS A FUNCTION OF $(Z_k, N_k(B'), N_k(F'), N_k(\sigma'))$

Z_k	$N_k(B')$	$N_k(F')$	$N_k(\sigma')$	$\mathcal{Q}_k$
e	≥ 2	—	—	ϕ
e	1	≥ 1	—	$\{\min F' \cap \psi_k\}$
e	1	0	≥ 1	$\{\alpha'_0, \alpha'_1\}$ where $\alpha'_0 = \min \sigma' \cap \psi_k$, $\alpha'_1 = \min \sigma_i \cap \psi_k \cap \{\alpha: \alpha \neq \alpha'_0\}$ where σ_i satisfies $\alpha'_0 \in \sigma_i$
e	0	≥ 2	—	ϕ
e	0	1	≥ 1	$\{\alpha'_0, \alpha'_0, \alpha'_2\}$ where $\alpha'_0 \in F' \cap \psi_k$, $\alpha'_1 = \min \sigma' \cap \psi_k$, $\alpha'_2 = \min \sigma_i \cap \psi_k \cap \{\alpha: \alpha \neq \alpha'_1\}$ where σ_i satisfies $\alpha'_1 \in \sigma_i$
e	0	0	≥ 2	If $\{\alpha'_0, \alpha'_1\} \subset \sigma_i$ for some i then $\mathcal{Q}_k = \{\alpha'_0, \alpha'_1\}$, else $\mathcal{Q}_k = \{\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3\}$, where $\alpha'_0 = \min \sigma' \cap \psi_k$ $\alpha'_1 = \min \sigma' \cap \psi_k \cap \{\alpha: \alpha \neq \alpha'_0\}$ $\alpha'_2 = \min \sigma_i \cap \psi_k \cap \{\alpha: \alpha \neq \alpha'_0\}$ where σ_i satisfies $\alpha'_0 \in \sigma_i$ $\alpha'_3 = \min \sigma_j \cap \psi_k \cap \{\alpha\ \alpha \neq \alpha'_1\}$ where σ_j satisfies $\alpha'_1 \in \sigma_j$
1	1	0	0	ϕ
1	0	1	0	ϕ
1	0	0	1	$\{\alpha'_1\}$ where $\alpha'_1 = \min \sigma_i \cap \psi_k \cap \{\alpha: \alpha \neq \alpha'_0\}$ where $\alpha'_0 \in \sigma' \cap \psi_k$ and σ_i satisfies $\alpha'_0 \in \sigma_i$
0	0	0	0	ϕ

with φ_k) is $\mathcal{F}_{k-1}$ measurable, where $\mathcal{F}_k = \sigma(Z_1,\cdots,Z_k, \mathcal{Q}_1,\cdots,\mathcal{Q}_k)$ is the σ-field generated by $Z_1,\cdots,Z_k, \mathcal{Q}_1,\cdots,\mathcal{Q}_k$.

Let ψ_k, $\psi_k \subset \{\alpha_1,\cdots,\alpha_W\}$, be the subset of packets which still remain to be transmitted at the end of $k-1$ slots. For example, $\psi_1 = \{\alpha_1 \cdots \alpha_W\}$. For any set A, $A \subset T$, let $N_k(A)$ be the number of packets in $A \cap \psi_k$.

An important property of our rule for specifying the sets $\mathcal{Q}_i$ is that, for any CRA and at the beginning of the kth slot, the conditional distribution of ψ_k given $\mathcal{F}_{k-1}$ can be parameterized by a set Γ. The set Γ consists of elements of the form $(\beta_1,\cdots,\beta_l, F, \sigma_1,\cdots,\sigma_m)$, where $\{\beta_1\},\cdots,\{\beta_l\}, F, \sigma_1,\cdots,\sigma_m$ are disjoint subsets of T. Corresponding to each $(\beta_1,\cdots,\beta_l, F, \sigma_1,\cdots,\sigma_m)$ in Γ is the following distribution of packets in T.

a) There is a packet at each of $\beta_1,\cdots,\beta_l$.
b) The distribution of packets in F is Poisson with intensity λ.
c) The distribution of packets in σ_i is Poisson with intensity λ conditioned on the event that there are at least two packets in σ_i.
d) The distribution of packets in distinct sets $\sigma_1,\cdots,\sigma_m$, F are independent.
e) All packets are contained in $\{\beta_1,\cdots,\beta_l\} \cup F \cup \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_m$.

The elements of Γ may be thought of as states of knowledge. At the beginning of the first slot, the state corresponds to $l = m = 0$ and $F = T$. Suppose after $k-1$ slots that the conditional distribution of ψ_k corresponds to $(\beta_1 \cdots \beta_l, F, \sigma_1,\cdots,\sigma_m)$ in Γ. Any set φ_k which a CRA can specify for slot k can be expressed as

$$\varphi_k = B' \cup F' \cup \sigma' \tag{1}$$

where

$$B' \subset \{1,\cdots,\beta_l\} \tag{2}$$

$$F' \subset F \tag{3}$$

$$\sigma' = \bigcup_{i=1}^{m} \sigma'_i \tag{4}$$

where $\sigma'_i \subset \sigma_i$.

We shall now assume that the set $\mathcal{Q}_k$ of packet locations revealed to the CRA at the end of the kth slot is specified as a function of $(Z_k, N_k(B'), N_k(F'), N_k(\sigma'))$ by Table I. Note that, except in the fourth case, enough packets are revealed to the CRA to account for the value of Z_k. The packets are revealed with the following priority: first packets from F' are revealed (if there are any) and then packets from σ' are revealed. Packets within these sets are given priority according to time of arrival. In addition, whenever one packet is revealed from a set σ_i, a second packet in σ_i is also revealed; then, σ_i is partitioned into two sets so that the locations of the two packets in the first set are known, and the packets in the second set are distributed at a Poisson point process with intensity λ.

Note that since the CRA already knows $N_k(B')$ at the beginning of slot k, after it also learns Z_k and $\mathcal{Q}_k$ it can determine which case of Table I was applied by counting the number of packets in $\mathcal{Q}_k$. Thus, not only do the packet locations in $\mathcal{Q}_k$ convey information, the number of packets in $\mathcal{Q}_k$ by itself conveys information. For example, even knowing that $\mathcal{Q}_k = \phi$ conveys information to the CRA.

It is not difficult to check that if the distribution of ψ_k corresponds to an element of Γ, then given $\mathcal{Q}_k$ and Z_k the conditional distribution of ψ_{k+1} again corresponds to an element of Γ. For example, suppose that the set φ_k of the form (1)–(4) enabled by a CRA in slot k satisfies $B' = \phi$. Suppose the CRA then learns that $Z_k = 1$ and is given the set $\mathcal{Q}_k$, which happens to contain one packet. This information implies that the second-from-last case in Table I is true. Packet α_0' has been successfully transmitted and α_1' joints the list of packets with known locations. Also, the distribution of packets in the unresolved portion of σ_i (where σ_i satisfies $\alpha_0', \alpha_1' \in \sigma_i$) is then Poisson with intensity λ. We see that the new state of knowledge for this example is

$$(\beta_1'', \cdots, \beta_{l+1}'', F'', \sigma_1'', \cdots, \sigma_{i-1}'', \sigma_{i+1}'', \cdots, \sigma_m''),$$

where $\beta_j'' = \beta_j$, for $1 \le j \le l$, $\beta_{l+1}'' = \alpha_1'$,

$$F'' = (F \cup (\sigma_i \cap \{\alpha : \alpha > \alpha_1'\})) \cap \varphi_i^c,$$

and $\sigma_j'' = \sigma_j \cap (\sigma')^c$. The new value of (l, m) is $(l+1, m-1)$. All other examples check out similarly, so by induction on k the conditional distribution of ψ_{k+1} given $\mathcal{F}_k$ corresponds to an element of Γ for all $k \ge 0$.

III. Domination of Increments

Now suppose a particular CRA is chosen. The algorithm can be extended to slots beyond τ by setting $\varphi_k = \varnothing$, for $k > \tau$. We denote the indicator random variable of an event A by I_A. Defined $Y_k = (Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)}) \in Z_+^3$ by $Y_k^{(1)} = l$ and $Y_k^{(2)} = m$, where $(\beta_1, \cdots, \beta_l, F, \sigma_1, \cdots, \sigma_m)$ is the state after the kth slot, and $Y_k^{(3)} = \sum_{j=1}^k I_{\{Z_j = 1\}}$, which is the number of packets successfully transmitted in the first k slots. Note that Y_k is $\mathcal{F}_k$-measurable. Define $Y_0 = (0,0,0)$, and let $X_k = Y_k - Y_{k-1}$.

Lemma: For all $k \ge 1$, $E[X_k \cdot V \mid \mathcal{F}_{k-1}] < 0.6126$, where $V = (0.3875, 0.5708, 1)$.

Assuming the lemma for a moment, we first establish the theorem. Clearly $Y_\tau \cdot V = (0,0,W) \cdot V = W$ so that $E[Y_\tau \cdot V] = E[W]$. On the other hand,

$$\begin{aligned} E[Y_\tau \cdot V] &= E\left[\sum_{k=1}^{\tau} X_k \cdot V\right] \\ &= E\left[\sum_{k=1}^{\infty} (X_k \cdot V) I_{\{k \le \tau\}}\right] \\ &= E\left[\sum_{k=1}^{\infty} E\left[X_k \cdot V I_{\{k \le \tau\}} \mid \mathcal{F}_{k-1}\right]\right] \\ &= E\left[\sum_{k=1}^{\infty} E\left[X_k \cdot V \mid \mathcal{F}_{k-1}\right] I_{\{k \le \tau\}}\right] \\ &\le E\left[\sum_{k=1}^{\infty} (0.6126) I_{\{k \le \tau\}}\right] \\ &= 0.6126 \sum_{k=1}^{\infty} P(k \le \tau) = 0.6126 E[\tau]. \end{aligned}$$

The fourth equality is justified by the fact that $\{k \le \tau\}$ is an $\mathcal{F}_{k-1}$ measurable event since $\{k \le \tau\} = \{\tau \le k-1\}$. Hence, $E[W]/E[\tau] \le 0.6126$, which proves the theorem.

Proof of Lemma:

Let $\varphi_k = \beta' \cup F' \cup \sigma'$ represent the set of stations designated by the CRA in slot k as in (1)–(4). Define $p_{a,b,c} = P(X_k = (a,b,c) \mid \mathcal{F}_{k-1})$.

Case $B' \ne \varnothing$: In this case

$$\begin{aligned} E&[X_k \cdot V \mid \mathcal{F}_{k-1}] \\ &= [(-1,0,1)p_{-1,0,1} + (2,-1,0)p_{2,-1,0} \\ &\quad + (1,0,0)p_{1,0,0}] \cdot V \\ &\le \max\{(-1,0,1) \cdot V, (2,-1,0) \cdot V, (1,0,0) \cdot V\} \\ &< 0.6126. \end{aligned}$$

Case $B' = \varnothing$: Let $x = \lambda \cdot \mu(F')$ where $\mu(F')$ is the Lebesgue measure of F'. Then $E[X_k \cdot V \mid \mathcal{F}_{k-1}] = \eta(x)$ is given by

$$\begin{aligned} \eta(x) = [&(1,-1,1)u_1 e^{-x} + (2,-1,0)u_{e,1}e^{-x} \\ &+ (4,-2,0)u_{e,2}e^{-x} + (0,0,1)u_0 x e^{-x} \\ &+ (3,-1,0)(1-u_0)xe^{-x} \\ &+ (0,1,0)(1-(1+x)e^{-x})] \cdot V, \end{aligned}$$

where

$$\begin{aligned} u_0 &= p_{0,0,1}e^x/x = P(N_k(\sigma') = 0), \\ u_1 &= p_{1,-1,1}e^x = P(N_k(\sigma') = 1), \\ u_{e,1} &= p_{2,-1,0}e^x \\ &= P(N_k(\sigma') \ge 2 \text{ and } N_k(\sigma_i') \ge 2, \\ &\qquad \text{where } \sigma_i' \text{ satisfies } \min(\sigma' \cap \psi_k) \subset \sigma_i') \\ u_{e,2} &= p_{4,-2,0}e^x \\ &= P(N_k(\sigma') \ge 2 \text{ and } N_k(\sigma_i') = 1, \\ &\qquad \text{where } \sigma_i' \text{ satisfies } \min(\sigma' \cap \psi_k) \subset \sigma_i'). \end{aligned}$$

Fact: $u_1 \le (1/2)$. To see this note that

$$\begin{aligned} u_1 &= P(N(\sigma') = 1) \\ &= \sum_{i=1}^m P(N(\sigma_i') = 1) \prod_{j \ne i} P(N(\sigma_j') = 0) \\ &\le \sum_{i=1}^m q_i \prod_{j \ne i} (1 - q_j) \triangleq S(q), \end{aligned}$$

where

$$q_i = P(N(\sigma_i') = 1), \qquad 1 \le i \le m$$

and

$$q = (q_1, \cdots, q_m).$$

Note that

a) $0 \le q_i \le (1/2)$ for $1 \le i \le m$;
b) S is a continuous function of q on $[0, 1/2]^m$;
c) S is affine in q_i for $1 \le i \le m$.

Therefore, S achieves its maximum value at one of the extreme points of $[0, 1/2]^m$. Straightforward evaluation at the extreme points of $[0, 1/2]^m$ yields the result $S \le 1/2$. This establishes the fact.

We wish to find an upper bound for $\eta(x)$, for every value of $x \ge 0$. Using the fact above, we have the following constraints:

$$\begin{gathered} 0 \le u_0 \le 1 \\ 0 \le u_1 \le \tfrac{1}{2} \\ 0 \le u_{e,1} \le 1. \\ 0 \le u_{e,2} \le 1 \\ u_0 + u_1 + u_{e,1} + u_{e,2} = 1. \end{gathered} \tag{5}$$

First simplify $\eta(x)$ to

$$\begin{aligned} \eta(x) = {} & u_0 x e^{-x}(0.4083) + u_1 e^{-x}(0.8167) \\ & + u_{e,1} e^{-x}(0.2042) + u_{e,2} e^{-x}(0.4084) \\ & + (0.5708)(1 - e^{-x}) + (0.0209) x e^{-x}. \end{aligned} \tag{6}$$

Under the constraints (5), it is easy to upper bound $\eta(x)$ for each fixed value of $x \ge 0$ since the right side of (6) is an affine function of $(u_0, u_1, u_{e,1} u_{e,2})$. This yields for $x \ge (0.8167/0.4083)$,

$$\eta(x) \le (0.5708)(1 - e^{-x}) + (0.4292) x e^{-x}.$$

For $0.4084/0.4083 \le x < 0.8167/0.4083$,

$$\eta(x) \le (0.5708) - (0.16245) e^{-x} + (0.22505) x e^{-x}.$$

For $0 \le x < 0.4084/0.4083$,

$$\eta(x) \le (0.5708) + (0.04175) e^{-x} + (0.0209) x e^{-x}.$$

These bounds imply that $\eta(x) < 0.6126$, for all $x \ge 0$. Indeed, this inequality need only be checked for $x = 0$, $x = -\infty$, and for values of x for which the derivative of the upper bound does not exist or is zero. This proves the lemma.

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