Applied Probability Trust (4 November 2004)

A QUEUE WITH SEMI-PERIODIC TRAFFIC

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Abstract

This paper analyzes the diffusion limit of a discrete time queueing system with constant service rate and connections that randomly enter and depart the system. Each connection generates periodic traffic while it is active, and a connection's lifetime has finite mean. This can model a TDMA system with constant bit rate connections. The diffusion scaling retains the semi-periodic behavior, allowing for short time (within one frame) and long time (multiple frames) analysis in the limit. Weak convergence of the cumulative arrival process and stationary buffer length distribution is proved. It is shown that the limit of the cumulative arrival process can be viewed as a discrete-time stationary increment Gaussian process interpolated by Brownian bridges. Bounds on the overflow probability of the limit queueing process as a function of the arrival rate and connection lifetime distribution are presented. Numerical and simulation results are presented for geometrically distributed connection lifetimes.

Keywords: semi-periodic traffic; diffusion limit; overflow; Gaussian process; queueing

AMS 2000 Subject Classification: Primary 60G15

Secondary 60F99;60G35

1. Introduction

Some types of real-time traffic sources, like digitized voice, generate data in a regular, periodic fashion. This paper considers such traffic in a FIFO (First In First Out)

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queueing system within the heavy traffic regime, where diffusion limits can be used to simplify the analysis.

Consider a time-slotted queueing system with constant service rate. Each connection enters the system, generates one packet every N slots (for some positive integer N) and departs after transmitting a random number of packets, independent of the other connections and of the time slot that the connection began generating packets. Packets are queued for transmission, and the transmission time of each packet is one slot. The numbers of new connections in distinct slots are assumed to be a mutually independent Poisson random variables. Assume that the connection lifetimes are identically distributed with finite mean.

We are interested in analyzing the performance of this system. This is a rather complex task, so we look at the diffusion limit of this system to simplify the analysis and get an estimate of the system behavior. A limit of the scaled cumulative arrival process is found as $N \to \infty$, uniformly over the scaled arrival rate. The semi-periodic nature of the network is reflected in the form of the autocorrelation function of the limit process. It is shown that the limit can be viewed as a discrete-time stationary increment Gaussian process interpolated by Brownian bridges. Bounds on the overflow probability of the limit queueing process as a function of the arrival rate and connection lifetime distribution are presented.

Our results complement Addie et al. [1] and Norros [8], which apply large deviation techniques to analyze queues with stationary Gaussian arrival process. In particular, the limit process in this paper is considered as an example in Norros [8]. Similar models are considered in Hajek [5] and Pazhyannur and Fleming [9]. The number of connections is fixed for each N in Hajek [5], as $N \to \infty$. A heavy traffic limit for fixed N is considered in Pazhyannur and Fleming [9] for a model with a Gaussian arrival process.

The paper is organized as follows. Section 2 presents the system model, the distribution of packet arrivals, and the buffering process. The proof of the convergence of the arrival process is carried out in Section 3 and Section 4 establishes some properties of the limit process. Section 5 establishes convergence of the normalized buffer length distribution. Section 6 presents bounds on the overflow probability of the limit queueing system. Finally, Section 7 presents numerical and simulation results for geometrically

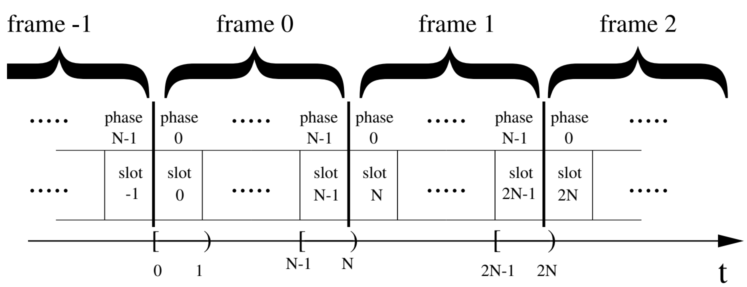


FIGURE 1: System model.

distributed connection lifetimes.

2. System model

This section presents the model used throughout the paper. For integer m, slot m consists of the time interval $[m, m+1)$. Group together N consecutive slots and call them a *frame*. Divide the slots into N equivalence classes, referred to as *phases*. Slots m and l are in the same phase if $m = l + nN$ for some integer n. Therefore, each frame consists of one slot from each of the N phases. Figure 1 depicts the relationship among frames, phases, slots, and continuous time.

Let L denote the number of packets sent by a connection (referred to as *connection lifetime* throughout the rest of the paper), and let its corresponding probability mass function be denoted by $f_L(l)$ for $l \geq 1$ with mean $\overline{L} < \infty$. Let $F_L^c(l) := P[L \geq l]$. The connection lifetimes are mutually independent.

Let $x_{k,j}$ denote the number of new connections in phase k of frame j, for $k = 0, \ldots, N-1$ and $j \geq 0$, and let $0 < \lambda_{min} \leq \frac{1}{\overline{L}}$. Assume that $x_{k,j}$ has the Poisson distribution with mean λ_N, such that $\lambda_N \in \left[\lambda_{min}, \frac{1}{\overline{L}}\right]$. The numbers of new connections in different slots are mutually independent. Since a connection generates 1 packet every N slots, if it enters the system in phase k of a frame, then it will keep sending 1 packet every frame (in the same phase) for a random number of frames. Clearly, the packet arrivals in different phases are independent of each other due to the independence between connections.

Denote the number of packets that arrive in slot m by $\tilde{a}_m$, and by $a_{k,j}$ the number of packets that arrive in phase k of frame j. This implies that $a_{k,j} = \tilde{a}_{k+Nj}$. Assume that packet arrivals occur only at the beginning of each slot.

Let m_1 and m_2 be integers with $m_1 \leq m_2$. A connection is said to have endpoints m_1 and m_2 if its first packet is generated in slot m_1 and its last packet is generated in slot m_2. If $m_2 - m_1 = N(l-1)$ for some positive integer l, then the number of connections with endpoints m_1 and m_2 has the Poisson distribution with mean $\lambda_N f(l)$. Otherwise, there are no connections with endpoints m_1 and m_2. From this observation, it is clear that the arrival process $(\tilde{a}_m)$ is time-reversible.

Section 2.1 gives the distribution of the number of packets arriving into the system in any particular slot, and Section 2.2 concerns the buffering process.

2.1. Distribution of packet arrivals

Since arrivals in different phases are independent, and the distribution of arrivals within each phase is the same, it suffices to obtain the distribution of arrivals within one phase.

Theorem 1. *For any $k = 0, \ldots, N-1$ and integer j, the distribution of $a_{k,j}$ is Poisson with mean $\lambda_N \overline{L}$. Furthermore, if j_1, j_2, k_1, k_2 are integers with $0 \leq k_1, k_2 \leq N-1$*

$$Cov(a_{k_1,j_1}, a_{k_2,j_2}) = \lambda_N \overline{L}\ g(|j_1 - j_2| + 1)\, \delta_{k_1,k_2} \tag{1}$$

where $g(l) := \dfrac{1}{\overline{L}} \displaystyle\sum_{n=l}^{\infty} F_L^c(n)$, and $\delta_{k_1,k_2} = 1$ if $k_1 = k_2$ and zero otherwise.

Proof. Fix k and let $j \in \mathbb{Z}$, then the number of packet arrivals in phase k of frame j corresponds to the number of active connections in the same frame and phase. This can be considered as the number of busy servers in an $M/G/\infty$ queue with Poisson arrivals with mean λ_N and i.i.d service times with probability mass function f_L. Then its steady state distribution has the following properties [7] :$a_{k,j} \sim Poisson\left(\lambda_N \overline{L}\right)$ and

$$\mathrm{Cov}\left(a_{k,j}, a_{k,j+n}\right) = \lambda_N E\left[(L - |n|)^+\right] = \lambda_N \sum_{l=|n|+1}^{\infty} F_L^c(l)$$

where x^+ denotes the positive part of x.

If $k_1 \neq k_2$, then by independence between phases, $\mathrm{Cov}\left(a_{k_1,j}, a_{k_2,j+n}\right) = 0$. Hence the proposition is proved. □

Section 3 will refer to these properties of the arrival process to obtain a diffusion limit for the model. It is pointed out in Krunz and Makowski [7] that the process could be long range dependent. In particular, if $E\left[L^2\right] = \infty$, then $\mathrm{Cov}\left(a_{k,j}, a_{k,j+n}\right)$ is not summable in n.

2.2. Buffer size

This section considers the buffering process. Let B_m be the cumulative number of arrivals from slot $-m$ up to slot -1, i.e., $B_m = \sum_{l=-m}^{-1} \tilde{a}_l$. Then the buffer size at time 0, denoted by Q_0^N, can be expressed as: $Q_0^N = \sup\{0, B_1 - 1, B_2 - 2, \ldots\}$.

Let $\tilde{A}_m$ denote the cumulative number of arrivals in $[0, m)$, i.e.,

$$\tilde{A}_m = \begin{cases} \displaystyle\sum_{l=0}^{m-1} \tilde{a}_l & m \geq 1 \\ 0 & m = 0 \end{cases} \tag{2}$$

Then by the reversibility of the arrival process

$$Q_0^N \stackrel{d}{=} \sup\{0, \tilde{A}_1 - 1, \tilde{A}_2 - 2, \ldots\} = \sup_{m \geq 0}\{\tilde{A}_m - m\} \tag{3}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

It will be helpful to group the arrivals by phases, so denote by $A_{k,j}$ the cumulative number of phase k arrivals in frames $\{0, 1, \ldots, j-1\}$, i.e., for $k = 0, \ldots, N-1$.

$$A_{k,j} = \begin{cases} \displaystyle\sum_{l=0}^{j-1} a_{k,l} & \forall j \geq 1 \\ 0 & j = 0 \end{cases} \tag{4}$$

Using (4) and the fact that packet arrivals occur only at the beginning of each slot, one can rewrite (2) as

$$\tilde{A}_t = \sum_{k=0}^{N-1} A_{k, l_k^N(t)} \qquad \forall t \geq 0$$

where $l_k^N(t) = \left\lfloor t + 1 - \frac{1+k}{N} \right\rfloor$, i.e., $l_k^N(t)$ denotes the number of complete phase k slots in $[0, t]$. Notice that this definition of $l_k^N(t)$ allows us to use any non-negative real number t instead of just integer values.

Section 3 considers a diffusion limit of this cumulative arrival process.

3. Diffusion limit

This section presents a diffusion limit for our model that allows for two time-scales, a short one corresponding to a single frame, and a long one that corresponds to multiple frames. Define on $t \geq 0$ and for $N \in \mathbb{N}$,

$$X_t^N = \frac{\tilde{A}_{Nt} - \lambda_N \overline{L} \lfloor Nt \rfloor}{\sqrt{\lambda_N \overline{L} N}} \tag{5}$$

For any $T > 0$ the process $X_t^N \in D[0,T]$, where $D[0,T]$ is the space of right continuous functions with left limits on the interval $[0,T]$. Denote by "$\Rightarrow$" weak convergence with respect to the Skorohod topology [2] on $D[0,T]$.

Let $\{X_t\}_{t\geq 0}$ be a stationary increment zero mean *a.s.* continuous Gaussian process with variance function:

$$\rho_t := E\left[X_t^2\right] = t + 2\sum_{j=1}^{\infty} g(j+1)(t-j)^+ \quad \forall\, t \geq 0 \tag{6}$$

Notice that the variance is piecewise linear with slope $1+2\sum_{j:1\leq j\leq k} g(j+1)$ on the interval $[k, k+1]$ where k is a non-negative integer. This slope approaches $\left(\frac{2}{\overline{L}}E\left[L^2\right] - 1\right)$ as $k \to \infty$, and as indicated in Section 2.1, if $E\left[L^2\right] = \infty$ then the process is long range dependent.

The covariance function of X is obtained from its stationary increments property and its variance:

$$\rho_{s,t} := E\left[X_s X_t\right] = \frac{1}{2}\left(\rho_s + \rho_t - \rho_{|s-t|}\right) \tag{7}$$

Since $\rho_t = t$ for $0 \leq t \leq 1$, the restriction $\{X_t\}_{0\leq t\leq 1}$ is a standard Brownian motion. Since X has stationary increments, $\{X_{t+\alpha} - X_\alpha\}_{0\leq t\leq 1}$ is also a standard Brownian motion for any $\alpha \geq 0$ fixed. Additional properties of X are given in Section 4.

Theorem 2. *For each $T > 0$, the random process $\left\{X_t^N\right\}_{0\leq t\leq T}$ converges weakly to the random process $\{X_t\}_{0\leq t\leq T}$ in $D[0,T]$ as $N \to \infty$, i.e., $X^N \Rightarrow X$.*

Proof. The proof is carried out in three steps. First, convergence of one dimensional distributions is shown. This result is then extended to finite dimensional distributions. It is then extended to weak convergence with respect to the Skorohod topology on $D[0,T]$.

Step 1

Let $t \geq 0$ be fixed throughout this step of the proof. Equation (5) can be rewritten as

$$X_t^N = \frac{1}{\sqrt{\lambda_N \overline{L} N}} \sum_{k=0}^{N-1} S_k^N(t)$$

where $S_k^N(t) = A_{k,l_k^N(Nt)} - l_k^N(Nt)\lambda_N \overline{L}$, i.e., $S_k^N(t)$ is the cumulative number of arrivals in phase k up to time Nt minus its mean.

Clearly, for fixed t, the random variables $S_k^N(t)$ $k = 0,\ldots,N-1$ are mutually independent since they correspond to arrivals in different phases. This representation of X_t^N makes it evident that it is the sum of N independent random variables.

The first and second moments, as well as an upper bound on the third moment, of $S_k^N(t)$ will be useful to show the desired convergence. The process S_k^N is centered so that $E\left[S_k^N(t)\right] = 0$. The second moment is given by

$$
\begin{aligned}
E\left[S_k^N(t)^2\right] &= \sum_{j_1=0}^{l_k^N(Nt)-1}\sum_{j_2=0}^{l_k^N(Nt)-1} \mathrm{Cov}(a_{k,j_1}, a_{k,j_2}) \\
&\overset{(a)}{=} \sum_{j=0}^{l_k^N(Nt)-1} \mathrm{Var}(a_{k,j}) + 2\sum_{j=1}^{l_k^N(Nt)-1} (l_k^N(Nt)-j)\mathrm{Cov}(a_{k,0}, a_{k,j}) \\
&\overset{(b)}{=} l_k^N(Nt)\lambda_N \overline{L} + 2\sum_{j=1}^{l_k^N(Nt)-1} (l_k^N(Nt)-j)\lambda_N \overline{L} g(j+1) \\
&\overset{(c)}{=} \begin{cases} \lambda_N \overline{L}\left(\lfloor t\rfloor + 1 + 2\displaystyle\sum_{j=1}^{\lfloor t\rfloor}(\lfloor t\rfloor + 1 - j)g(j+1)\right) & if \quad t - \lfloor t\rfloor \geq \frac{k+1}{N} \\ \lambda_N \overline{L}\left(\lfloor t\rfloor + 2\displaystyle\sum_{j=1}^{\lfloor t\rfloor}(\lfloor t\rfloor - j)g(j+1)\right) & else \end{cases}
\end{aligned}
\tag{8}
$$

where (a) follows from the stationarity of the arrivals, (b) follows from (1), and (c) follows from the fact that $l_k^N(Nt) = \lfloor t\rfloor + 1$ if $t - \lfloor t\rfloor \geq \frac{k+1}{N}$ and $l_k^N(Nt) = \lfloor t\rfloor$

otherwise. Note that if $a, b \geq 0$, then $|a-b|^3 \leq \max\{a^3, b^3\} \leq a^3+b^3$. Thus

$$
\begin{aligned}
E\left[\left|S_k^N(t)\right|^3\right] &\overset{(a)}{\leq} E\left[\left(\sum_{m=1}^{l_k^N(Nt)} mZ_m\right)^3 + \left(\sum_{m=1}^{l_k^N(Nt)} m\mu_m\right)^3\right] \\
&\leq (l_k^N(Nt))^3 E\left[\left(\sum_{m=1}^{l_k^N(Nt)} Z_m\right)^3 + \left(\sum_{m=1}^{l_k^N(Nt)} \mu_m\right)^3\right] \\
&\overset{(b)}{\leq} (l_k^N(Nt))^3 \left(\left(1+\sum_{m=1}^{l_k^N(Nt)} \mu_m\right)^3 + \left(\sum_{m=1}^{l_k^N(Nt)} \mu_m\right)^3\right) \\
&\leq (l_k^N(Nt))^3 \left(1+2\sum_{m=1}^{l_k^N(Nt)} \mu_m\right)^3 \\
&\leq l_k^N(Nt)^3 \left(1+2\lambda_N \overline{L}\, l_k^N(Nt)\right)^3 \overset{(c)}{\leq} \lceil t\rceil^3 \left(1+2\lambda_N \overline{L}\lceil t\rceil\right)^3
\end{aligned} \tag{9}
$$

where Z_m is the number of connections that contribute m arrivals counted in $S_k^N(t)$, and $\mu_m = E[Z_m]$. The Z_m's are mutually independent Poisson random variables, and $\sum_{m=1}^{l_k^N(Nt)} m\mu_m = \lambda_N \overline{L}\, l_k^N(Nt)$. Inequality (a) follows because both Z_m and μ_m are nonnegative, (b) follows because for a Poisson random variable with mean μ, its third moment equals $\mu^3+3\mu^2+\mu < (1+\mu)^3$, and (c) follows because $l_k^N(Nt) \leq \lceil t\rceil$.

It will now be shown that the random variable X_t^N converges in distribution to a Gaussian random variable. For each integer $N \geq 1$ and $k = 0, \ldots, N-1$ let $Y_{N,k} = \frac{1}{\sqrt{\lambda_N \overline{L} N}} S_k^N(t)$. Then $E[Y_{N,k}] = 0$. Also,

$$
\begin{aligned}
\sum_{k=0}^{N-1} E\left[Y_{N,k}^2\right] &= \frac{1}{\lambda_N \overline{L} N} \sum_{k=0}^{N-1} E\left[S_k^N(t)^2\right] \\
&\overset{(a)}{=} \frac{\lfloor Nt\rfloor - N\lfloor t\rfloor}{N}\left(\lfloor t\rfloor + 1 + 2\sum_{j=1}^{\lfloor t\rfloor}(\lfloor t\rfloor + 1 - j)g(j+1)\right) + \\
&\quad \frac{N-(\lfloor Nt\rfloor - N\lfloor t\rfloor)}{N}\left(\lfloor t\rfloor + 2\sum_{j=1}^{\lfloor t\rfloor - 1}(\lfloor t\rfloor - j)g(j+1)\right) \\
&\overset{(b)}{=} \frac{\lfloor Nt\rfloor}{N} + 2\sum_{j=1}^{\lfloor t\rfloor}\left(\frac{\lfloor Nt\rfloor}{N} - j\right)g(j+1) \\
&\to t + 2\sum_{j=1}^{\lfloor t\rfloor}(t-j)g(j+1) = \rho_t
\end{aligned} \tag{10}
$$

where (a) follows the fact that $t - \lfloor t \rfloor \geq \frac{k+1}{N}$ for the first $\lfloor Nt \rfloor - N\lfloor t \rfloor$ terms in the sum and from (8). Equality (b) follows from grouping terms. Notice that $\rho_t < \infty$.

Now, for any $\epsilon > 0$,

$$\sum_{k=0}^{N-1} E\left[|Y_{N,k}|^2; |Y_{N,k}| > \epsilon\right] \leq \sum_{k=0}^{N-1} E\left[\frac{|Y_{N,k}|^3}{\epsilon}\right] = \frac{1}{\epsilon\left(\lambda_N \overline{L} N\right)^{\frac{3}{2}}} \sum_{k=0}^{N-1} E\left[\left|S_k^N(t)\right|^3\right]$$

$$\stackrel{(a)}{\leq} \frac{1}{\epsilon\left(\lambda_N \overline{L} N\right)^{\frac{3}{2}}} \sum_{k=0}^{N-1} \lceil t \rceil^3 \left(1 + 2\lambda_N \overline{L} \lceil t \rceil\right)^3 \stackrel{(b)}{\leq} N \frac{\lceil t \rceil^3 \left(1 + 2\lceil t \rceil\right)^3}{\epsilon\left(\lambda_N \overline{L} N\right)^{\frac{3}{2}}} \stackrel{(c)}{\leq} \frac{\lceil t \rceil^3 \left(1 + 2\lceil t \rceil\right)^3}{\epsilon\left(\lambda_{min} \overline{L}\right)^{\frac{3}{2}}} \frac{1}{\sqrt{N}} \to 0$$

where (a) follows from (9), (b) follows because $0 < \lambda_N \overline{L} \leq 1$, and (c) because $0 < \lambda_{min} \leq \lambda_N$.

Hence, the sequence $\{Y_{N,k}\}$ satisfies the conditions for the Lindeberg-Feller theorem [4], and so

$$X_t^N = \sum_{k=0}^{N-1} Y_{N,k} \Rightarrow X_t \sim N(0, \rho_t)$$

Step 2

Now, for any $n \geq 1$, $t_1, \ldots, t_n \geq 0$, the vector $\hat{X}_n^N = \left(X_{t_1}^N, \ldots, X_{t_n}^N\right)$ is zero mean. For any real valued n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$, the procedure used in Step 1 can be applied to represent the random variable $\gamma \cdot \hat{X}_n^N$ as a sum of N independent random variables which converges to a Gaussian random variable. Furthermore, the variance of $\gamma \cdot \hat{X}_n^N$ converges to that of $\gamma \cdot \hat{X}_n$, where $\hat{X}_n = (X_{t_1}, \ldots, X_{t_n})$ is an n-dimensional zero mean Gaussian vector whose covariance matrix components are $E\left[X_{t_i} X_{t_j}\right] = \rho_{t_i, t_j}$. Then, by the Cramér-Wold device [4], $\left(X_{t_1}^N, \ldots, X_{t_n}^N\right) \Rightarrow (X_{t_1}, \ldots, X_{t_n})$. So the finite-dimensional distributions converge as claimed.

Step 3

The result is now extended to weak convergence of measures on $D[0, T]$. The following upper bound on the second moment of X_t^N is used to carry out this step.

$$\begin{aligned} E\left[\left(X_t^N\right)^2\right] &\stackrel{(a)}{=} \frac{1}{\lambda_N \overline{L} N} \sum_{k=0}^{N-1} E\left[S_k^N(t)^2\right] \\ &\stackrel{(b)}{=} \frac{\lfloor Nt \rfloor}{N} + 2\sum_{j=1}^{\lfloor t \rfloor} \left(\frac{\lfloor Nt \rfloor}{N} - j\right) g(j+1) \leq t + 2\sum_{j=1}^{\lfloor t \rfloor} (t-j) g(j+1) \\ &\stackrel{(c)}{\leq} t + 2\sum_{j=1}^{\lfloor t \rfloor} (t-j) \stackrel{(d)}{\leq} t(1+2T) := F(t) \end{aligned} \tag{11}$$

where (a) follows from independence of phases and $E[S_k^N(t)] = 0$, (b) follows from (10), (c) follows from the fact $g(l) \leq g(1) = 1$ for $l \geq 1$, and (d) follows from the fact that $\lfloor t \rfloor \leq T$.

Consider any $0 \leq r < s < t \leq T$. Also assume that $\lfloor Ns \rfloor - \lfloor Nr \rfloor$ and $\lfloor Nt \rfloor - \lfloor Ns \rfloor$ are both nonzero, for otherwise one of the increments

$X^N(s) - X^N(r)$ or $X^N(t) - X^N(s)$ is identically zero. If follows that

$$\lfloor Ns \rfloor - \lfloor Nr \rfloor \leq N(t-r) \text{ and } \lfloor Nt \rfloor - \lfloor Ns \rfloor \leq N(t-r) \tag{12}$$

On the one hand, if $0 \leq \lfloor Nt \rfloor - \lfloor Nr \rfloor \leq N$, then the increments $X_s^N - X_r^N$ and $X_t^N - X_s^N$ are independent since they correspond to arrivals in different phases. So, in this case, if $\mu > 0$,

$$\begin{aligned}
P\left[|X_s^N - X_r^N| \wedge |X_t^N - X_s^N| \geq \mu\right] &= P\left[|X_s^N - X_r^N| \geq \mu\right] P\left[|X_t^N - X_s^N| \geq \mu\right] \\
&\overset{(a)}{\leq} \frac{1}{\mu^2} E\left[|X_s^N - X_r^N|^2\right] \frac{1}{\mu^2} E\left[|X_t^N - X_s^N|^2\right] \overset{(b)}{\leq} \frac{1}{\mu^4} F\left(\frac{\lfloor Ns \rfloor - \lfloor Nr \rfloor}{N}\right) F\left(\frac{\lfloor Nt \rfloor - \lfloor Ns \rfloor}{N}\right) \\
&\overset{(c)}{\leq} \frac{1}{\mu^4} \left(F(t-r)\right)^2 \overset{(d)}{=} \frac{1}{\mu^4} \left(F(t) - F(r)\right)^2
\end{aligned}$$

where (a) follows from Chebyshev's inequality, (b) follows from the stationarity of the increments of $X_{i/N}^N$ and from (11), (c) follows from (12) and from $F(\cdot)$ being increasing, and (d) follows from the linearity of $F(\cdot)$.

On the other hand, if $\lfloor Nt \rfloor - \lfloor Nr \rfloor > N$, the increments are not independent. In this case, if $\mu > 0$

$$\begin{aligned}
P\left[|X_s^N - X_r^N| \wedge |X_t^N - X_s^N| \geq \mu\right] &\overset{(a)}{\leq} P\left[|X_s^N - X_r^N| \geq \mu\right] \overset{(b)}{\leq} \frac{1}{\mu^2} E\left[|X_s^N - X_r^N|^2\right] \\
&\overset{(c)}{\leq} \frac{1}{\mu^2} F\left(\frac{\lfloor Ns \rfloor - \lfloor Nr \rfloor}{N}\right) \overset{(d)}{\leq} \frac{1}{\mu^2} F(t-r) \overset{(e)}{\leq} \frac{1}{\mu^2} \left(F(t-r)\right)^2 \overset{(f)}{=} \frac{1}{\mu^2} \left(F(t) - F(r)\right)^2
\end{aligned}$$

where (a) follows from the fact that the probability of the intersection of two events is smaller than the probability of either one of them. Then (b) follows from Chebyshev's inequality, (c) follows from the stationarity of the increments of $X_{i/N}^N$ and from (11), (d) follows from (12) and $F(\cdot)$ being increasing, (e) follows from the fact that $F(t-r)$ is greater than 1, and (f) follows from the linearity of $F(\cdot)$.

Hence, for any $0 \leq r < s < t \leq T$,

$$P\left[|X_s^N - X_r^N| \wedge |X_t^N - X_s^N| \geq \mu\right] \leq \frac{1}{\mu^2 \wedge \mu^4} \left(F(t) - F(r)\right)^2$$

Also, for any $\delta > 0$ the increment $X_T - X_{T-\delta} \sim N(0, \rho_\delta)$, therefore $X_T - X_{T-\delta} \Rightarrow 0$ as $\delta \to 0$.

Then, by [2, Theorem 13.5] the process $X_t^N \Rightarrow X_t$ where X_t is a stationary increments Gaussian process with variance as indicated by (6). The theorem has been proved. □

4. Properties of the limit process

This section discusses three properties of the limit process X_t. Section 4.1 describes its derivative, Section 4.2 its distribution between integer time intervals, and Section 4.3 a law of large numbers.

4.1. Derivative of the limit process

Although the derivative process $\dot{X}_t$ exists only as a generalized Gaussian random process, its autocorrelation function gives insight about X_t.

$$E\left[\dot{X}_{t_1}\dot{X}_{t_2}\right] = \frac{\partial^2 \rho_{t_1,t_2}}{\partial t_1 \partial t_2} \overset{(a)}{=} -\frac{1}{2}\frac{\partial^2 \rho_{|t_1-t_2|}}{\partial t_1 \partial t_2} = \sum_{n=-\infty}^{\infty} g\left(|n|+1\right)\delta(t_1 - t_2 - n)$$

where (a) follows from the stationarity of the increments of X_t. Thus, formally, $\dot{X}_{t_1}$ is independent of $\dot{X}_{t_2}$ unless $t_1 - t_2$ is an integer, and $E\left[\dot{X}_{t_1}\dot{X}_{t_1+n}\right]$ is proportional to $g\left(|n|+1\right)$. This property mirrors the properties of the original process, specifically the autocorrelation function (1). It also is consistent with the fact that $\{X_{t+\alpha} - X_t\}_{0\le t\le 1}$ is a standard Brownian motion for $\alpha \ge 0$ fixed.

4.2. Distribution of the process between integer time intervals

The process $\{X_t\}_{t\in\mathbb{R}+}$ can be viewed as an interpolation of $(X_n)_{n\in\mathbb{N}}$, where Brownian bridges are used for the interpolation. A Brownian bridge is a stationary increment zero mean *a.s.* continuous Gaussian process $\{B_t\}_{0\le t\le 1}$ with covariance function $E\left[B_s B_t\right] = min(s,t)\left(1 - max(s,t)\right)$. See Karatzas and Shreve [6] for more details. This characterization of X_t is helpful to obtain bounds on the overflow probability of the limit queueing system since if B is a Brownian bridge then [6] for $\alpha + \beta \ge 0$ and $\beta > 0$

$$P\left[\max_{0\le t\le 1}\{B_t - \alpha t\} \le \beta\right] = 1 - \exp\left(-2\beta(\alpha+\beta)\right) \tag{13}$$

Define for $j \in \mathbb{N}$ and for $0 \leq t \leq 1$

$$\eta_t^j := X_{t+j} - [(1-t)X_j + tX_{j+1}]$$

Theorem 3. *For each $j \in \mathbb{N}$, η^j is a Brownian bridge and is independent of $(X_n)_{n\in\mathbb{N}}$. Furthermore, Cov $\left(\eta_s^j, \eta_t^{j+n}\right) = g\,(|n|+1)\,min(s,t)\,(1-max(s,t))$.*

Proof. Since X is a mean zero Gaussian process, so is η^j. Hence the mean and covariance function of η^j specify its distribution completely. Let $j \in \mathbb{N}$ and suppose $0 \leq s \leq t \leq 1$, then

$$\begin{aligned}\mathrm{Cov}(\eta_s^j, \eta_t^j) &= \rho_{s+j,t+j} - (1-t)\rho_{s+j,j} - t\rho_{s+j,j+1} - (1-s)\rho_{j,t+j} + (1-s)(1-t)\rho_j \\ &\quad +(1-s)t\rho_{j,j+1} - s\rho_{j+1,j+t} + s(1-t)\rho_{j+1,j} + st\rho_{j+1} = (s-st)\end{aligned}$$

using (7) and simple algebraic manipulation. Similarly, $\mathrm{Cov}(\eta_s^j, \eta_t^j) = (t-st)$ if $t < s$. Therefore $\mathrm{Cov}(\eta_s^j, \eta_t^j) = min(s,t)\,(1-max(s,t))$, and hence η^j is a Brownian bridge.

Now, to show the claimed independence it is only needed to check that $\mathrm{Cov}(\eta_t, X_n) = 0$ for any $j, n \in \mathbb{N}$ and $0 \leq t \leq 1$ since $(X_n)_{n\in\mathbb{N}}$ and $\left(\eta^j\right)_{j\in\mathbb{N}}$ are jointly Gaussian.

$$\mathrm{Cov}(\eta_t^j, X_n) = \rho_{t+j,n} - (1-t)\rho_{j,n} - t\rho_{j+1,n} = 0$$

using (7) and simple algebraic manipulation. Therefore η^j is independent of $(X_n)_{n\in\mathbb{N}}$.

Let $j \in \mathbb{N}$, $n \in \mathbb{Z}$ and $0 \leq s,t \leq 1$, then

$$\begin{aligned}\mathrm{Cov}\left(\eta_s^j, \eta_t^{j+n}\right) &= \mathrm{Cov}\left(\eta_s^j, X_{t+j+n} - [(1-t)X_{j+n} + tX_{j+n+1}]\right) \\ &\stackrel{(a)}{=} \rho_{s+j,t+j+n} - (1-s)\rho_{j,t+j+n} - s\rho_{j+1,t+j+n} \\ &\stackrel{(b)}{=} \begin{cases} g(n+1)s(1-t) & n \geq 0 \;\; and \;\; t \geq s \\ g(n+1)t(1-s) & n \geq 0 \;\; and \;\; t < s \\ g(-n+1)s(1-t) & n < 0 \;\; and \;\; t \geq s \\ g(-n+1)t(1-s) & n < 0 \;\; and \;\; t < s \end{cases}\end{aligned}$$

where (a) follows from independence between η^j and $(X_n)_{n\in\mathbb{N}}$, and (b) from (7) and simple algebraic manipulation.

Therefore $\mathrm{Cov}\left(\eta_s^j, \eta_t^{j+n}\right) = g\,(|n|+1)\,min(s,t)\,(1-max(s,t))$, and hence the proposition has been proved. □

4.3. A strong law of large numbers for X

The following theorem shows that, under a mild condition on ρ, the process X satisfies a strong law of large numbers (SLLN).

Theorem 4. *Suppose that for some positive constants K and ϵ, $\rho_t \leq Kt^{2-\epsilon}$ for all t. Then $\lim_{t\to\infty} X_t/t = 0$ almost surely.*

Proof. Since X and $-X$ have the same distribution, it is enough to prove that $\limsup_{t\to\infty} X_t/t = 0$ almost surely. The key idea of the proof is to treat X at integer times, and then appeal to the fact that over any interval of the form $[j, j+1]$ for an integer j, X can be written as the sum of the linear interpolation between X_j and X_{j+1} plus the Brownian bridge $\eta^j(t)$. Thus, for any $t \in [j, j+1]$,

$$X_t \leq \max\{X_j, X_{j+1}\} + \mathcal{M}(\eta^j), \tag{14}$$

where $\mathcal{M}(\eta^j)$ denotes the (random) maximum value of the Brownian bridge η^j. Let $\alpha > 0$ and define $\tau_0 = \sup\{t : X_t \geq \alpha(2t+1)\}$. To complete the proof it suffices to prove that $P[\tau_0 < \infty] = 1$. Define two more random times by $\tau_1 = \sup\{j : X_j \geq \alpha j\}$ and $\tau_2 = \sup\{j : \mathcal{M}(\eta^j) \geq \alpha j)\}$. If $t \geq \max\{\tau_1, \tau_2\} + 1$, then $X_{\lfloor t \rfloor} < \alpha \lfloor t \rfloor \leq \alpha t$, $X_{\lceil t \rceil} < \alpha \lceil t \rceil \leq \alpha(t+1)$, and $\mathcal{M}(\eta^{\lfloor t \rfloor}) < \alpha \lfloor t \rfloor \leq \alpha t$. Therefore, by (14), $X_t < \alpha(2t+1)$, for any t with $t \geq \max\{\tau_1, \tau_2\} + 1$. Consequently, $\tau_0 \leq \max\{\tau_1, \tau_2\} + 1$. Since for an integer j, X_j is Gaussian with mean zero and variance ρ_j,

$$P[X_j \geq \alpha j] \leq \exp(-\frac{\alpha^2 j^2}{2\rho_j}) \leq \exp(-\frac{\alpha^2 j^\epsilon}{2K}).$$

Since η^j is a standard Brownian bridge for each j, $P[\mathcal{M}(\eta^j) \geq \alpha j] = \exp(-2(\alpha j)^2)$, by (13). Combining the above and using a union bound yields:

$$\begin{aligned} P[\tau_0 \geq t] &\leq P[\tau_1 \geq t-1] + P[\tau_2 \geq t-1] \\ &\leq \sum_{j=t-1}^{\infty} P[X_j \geq \alpha j] + \sum_{j=t-1}^{\infty} P[\mathcal{M}(\eta^j) \geq \alpha j] \\ &= \sum_{j=t-1}^{\infty} \left[\exp(-\frac{\alpha^2 j^\epsilon}{2K}) + \exp(-2(\alpha j)^2) \right] \end{aligned} \tag{15}$$

so that $P[\tau_0 \geq t] \to 0$ as $t \to \infty$. Thus, $P[\tau_0 < \infty] = 1$. In addition, (15) provides an upper bound on the tail of the distribution of τ_0. $\square$

Let $\theta > 0$ and define the random variable Q_0 by $Q_0 = \sup_{t\geq 0}\{X_t - \theta t\}$. The following corollary is an immediate consequence of Theorem 4.

Corollary 1. *Under the assumption of Theorem 4, Q_o is finite with probability one.*

5. Convergence of buffer distribution

The following assumption will be invoked.

Assumption A *The probability mass function f of the connection length L satisfies $f(i) \leq Di^{-(2+\epsilon)}$ for all $i \geq 1$, for some positive constants D and ϵ.*

Assumption A insures that $\overline{L} < \infty$, but if $\epsilon \leq 1$ it permits long range dependence, meaning $E[L^2] = +\infty$ is possible. The proof of the following lemma is left to the reader.

Lemma 1. *Under Assumption A, there exist constants K_1, K_2, and K_3 so that $F^c(n) \leq K_1 n^{-(1+\epsilon)}$, $g(l) \leq K_2 l^{-\epsilon}$, and $\rho_t \leq K_3 t^{2-\epsilon}$.*

Theorem 5. *Let Assumption A hold, and assume that as $N \to \infty$, $\lambda_N \overline{L} \to 1$ in such a way that $\frac{(1-\lambda_N\overline{L})\sqrt{N}}{\sqrt{\lambda_N\overline{L}}} \to \theta$, for a constant $\theta > 0$. Then,*

$$\frac{Q_0^N}{\sqrt{\lambda_N \overline{L} N}} \Rightarrow Q_0$$

The remainder of this section provides a proof of Theorem 5. Equation (3) can be rewritten using (5) as follows:

$$\begin{aligned} Q_0^N &= \sup_{m\geq 0}\{\tilde{A}_m - m\} = \sup_{m\geq 0}\left\{\sqrt{\lambda_N\overline{L}N}X^N_{(\frac{m}{N})} - m(1-\lambda_N\overline{L})\right\} \\ &= \sqrt{\lambda_N\overline{L}N}\sup_{m\geq 0}\left\{X^N_{(\frac{m}{N})} - \frac{m(1-\lambda_N\overline{L})}{\sqrt{\lambda_N\overline{L}N}}\right\} \overset{(a)}{=} \sqrt{\lambda_N\overline{L}N}\sup_{t\geq 0}\left\{X^N_t - \frac{\lfloor Nt\rfloor}{\sqrt{N}}\frac{1-\lambda_N\overline{L}}{\sqrt{\lambda_N\overline{L}}}\right\} \end{aligned}$$

where (a) follows from a change of variables $t = \frac{m}{N}$ and the fact that the process is constant over the interval $[\frac{m}{N}, \frac{m+1}{N})$.

The convergence guaranteed by Theorem 2 holds only on a finite interval of normalized time, $[0, T]$, where T can be arbitrarily large, but not dependent on N. Thus, define the random variables

$$\begin{aligned}
Q_0^{N,T} &= \sup_{0\le m\le NT}\{\widetilde{A}_m - m\} = \sqrt{\lambda_N \overline{L} N} \sup_{0\le t\le T}\left\{X_t^N - \frac{\lfloor Nt\rfloor}{\sqrt{N}}\frac{1-\lambda_N\overline{L}}{\sqrt{\lambda_N\overline{L}}}\right\} \\
Q_0^T &= \max_{0\le t\le T}\{X_t - \theta t\}
\end{aligned}$$

Theorem 2 and the continuous mapping theorem [2] imply that for T fixed,

$$\frac{Q_0^{N,T}}{\sqrt{\lambda_N\overline{L}N}} \Rightarrow Q_0^T$$

Hence, to complete the proof of Theorem 5, it suffices to establish the following two statements, for some constant N_o:

$$P[Q_0^T = Q_0] \quad \to 1 \qquad \text{as } T\to\infty \tag{16}$$

$$\inf_{N_o\le N<\infty} P[Q_0^{N,T} = Q_0^N] \quad \to 1 \qquad \text{as } T\to\infty \tag{17}$$

Theorem 4 and its proof will be used to prove (16), and to guide the proof of (17). Select α so that $2\alpha<\theta$, let τ_0 be the random time defined in the proof of Theorem 4, and let t_o be a constant large enough that $\theta t \ge \alpha(2t+1)$ for $t\ge t_o$. Then $X_t - \theta t\le 0$ for $t\ge \max\{\tau_o, t_o\}$. Consequently, for any $T\ge t_o$, $P[Q_0^T = Q_0] \ge P[\tau_0\le T]$, and $P[\tau_0\le T]$ converges to one as $T\to\infty$. This establishes (16). It remains to prove (17). The proof is achieved by showing that X^N satisfies the upper-bound half of a SLLN, with the rate of convergence uniform in N. That will be done by paralleling the proof of Theorem 4. Equation (15) provides a bound on the tail of the distribution of τ_0, so the idea will be to establish a similar bound for the processes X^N that holds uniformly in N. As in the proof of Theorem 4, the first step is to bound the process at integer times by using uniform exponential bounds on Possion random variables, similar to the exponential bounds on the Gaussian distribution used in the proof of Theorem 4. The second step is to consider bridge processes to handle the fluctuations within frames. The following lemma takes care of the first step.

Lemma 2. *Given any $\alpha>0$, there exist T_o, N_o, and $b_{t,\alpha}$ for integers $t\ge T_o$, so that:*

$$\begin{aligned}
&P[X_t^N \ge 3\alpha t] \quad \le \quad b_{t,\alpha} \quad \textit{for} \;\; t\ge T_o,\; N\ge N_o \\
&\sum_{t=T_o}^{\infty} b_{t,\alpha} < \infty
\end{aligned}$$

Lemma 2 will be proved with the help of some lemmas, stated next. The following function ψ plays a very useful role in the development of exponential bounds for binomial and Poisson random variables (see [10]): For $\lambda \in [-1, +\infty)$,

$\psi(\lambda) = 2h(1+\lambda)/\lambda^2$, with

$h(\lambda) = \lambda(\log \lambda - 1) + 1$. The function ψ is strictly positive and strictly decreasing on the interval $[-1, +\infty)$, with $\psi(-1) = 2$ and $\psi(0) = 1$. Also, $\lambda\psi(\lambda)$ is strictly increasing in λ over the interval $[-1, +\infty)$. The following lemma is a useful arrangement of the Chernoff inequality applied to Poisson random variables.

Lemma 3. *Let V be a Poisson random variable with mean $\mu > 0$. Then*

$$P[V - E[V] \geq c] \leq \exp\left(-\frac{c^2}{2\mu}\psi\left(\frac{c}{\mu}\right)\right) \qquad \text{for } c \geq 0 \tag{18}$$

Proof. Since the log moment generating function of V is given by $\log E[\exp(s(V - E[V])] = \mu(e^s - 1 - s)$, the Chernoff inequality implies that for $c \geq 0$,

$$\begin{aligned} P[V - E[V] \geq c] &\leq \min_{s \geq 0} \exp(-sc + \mu(e^s - 1 - s)) \\ &= \exp\left(-\frac{c^2}{2\mu}\psi\left(\frac{c}{\mu}\right)\right), \end{aligned}$$

which proves the lemma. □

Lemma 4. *Let $V = \sum_{i=0}^{i_o} iV_i$, where V_i is a Poisson random variable with mean μ_i, i_o is a positive integer, and the V_i's are mutually independent. Suppose $E[V] > 0$, and let $\kappa = \sum_{i=0}^{i_o} i^2\mu_i$. Then,*

$$P[V - E[V] \geq c] \leq \exp\left(-\frac{c^2}{2\kappa}\psi\left(\frac{ci_o}{\kappa}\right)\right) \qquad \text{for } c \geq 0 \tag{19}$$

Furthermore, the right side of (19) is monotonically increasing in κ.

Proof. If $1 \leq i \leq i_0$, then $e^{is} - 1 - is = \sum_{k=2}^{\infty} \frac{(is)^k}{k!} \leq \frac{i^2}{i_o^2}(e^{i_o s} - 1 - i_o s)$. Hence

$$\log E[\exp(s(V - E[V])] = \sum_{i=1}^{i_o} \mu_i(e^i s - 1 - is) \leq \frac{\kappa}{i_o^2}(e^{i_o s} - 1 - i_o s)$$

Thus, the log moment generating function for the centered compound Poisson random variable $V - E[V]$ is less than or equal to the log moment generating function for i_0 times a centered Poisson random variable with mean κ/i_o^2. So we can replace c by c/i_o and μ by κ/i_o^2 in the bound of Lemma 3 to yield the bound of Lemma 4. The monotonicity follows from the monotonicity of $\lambda\psi(\lambda)$. □

Consider an interval of t frames, for some integer t. Let $r(i,t)$ denote the mean number of connections contributing i packets to one of the phases during the interval, for $1 \leq i \leq t$. Such connections (1) either start or end during the interval, but not both, (2) start and end during the interval, or (3) start before the interval and end after the interval. Accounting for the mean number of each of these types of connections yields:

$$r(i,t) = \lambda_N \left[2 \sum_{j=i+1}^{\infty} f(j) + (t-i+1)f(i) + I_{\{i=t\}} \sum_{j=t+2}^{\infty} (j-t-1)f(j) \right] \tag{20}$$

Comparisons with integrals show that Assumption A implies the following bounds for some constants D_1, D_2, D_3, and D_4:

$$r(i,t) \leq \begin{cases} D_1 t i^{-(2+\epsilon)} & 1 \leq i < t \\ D_2 t^{-\epsilon} & i = t \end{cases} \tag{21}$$

$$\sum_{j=\lfloor t^\epsilon \rfloor}^{t} r(j,t) \leq D_3 t^{-\epsilon} \tag{22}$$

$$\sum_{j=1}^{t} i^2 r(i,t) \leq D_4 t^{2-\epsilon} \tag{23}$$

Connections contributing one or more packets during the interval of frames 0 through $t-1$ are classified into two types as follows. A connection that generates between 1 and $\lfloor t^\epsilon \rfloor$ packets during the interval is called a type one connection. A connection that generates between $\lfloor t^\epsilon \rfloor + 1$ and t packets during the interval is called a type two connection. We can then write

$$X_t^N = \frac{Z_1 - E[Z_1] + Z_2 - E[Z_2]}{\sqrt{\lambda_N L N}}$$

where Z_i is the total number of packets contributed by type i connections during the interval of t frames. Since $\lambda_N \overline{L} \to 1$ as $N \to \infty$, it can be assumed that N_o is chosen large enough that $3\sqrt{\lambda_N \overline{L}} \geq 2$ for all $N \geq N_o$. Then

$$\begin{aligned} P[X_t^N \geq 3\alpha t] &\leq P[Z_1 - E[Z_1] + Z_2 - E[Z_2] \geq 2\alpha\sqrt{N}t] \\ &\leq P[Z_1 - E[Z_1] \geq \alpha\sqrt{N}t] + P[Z_2 - E[Z_2] \geq \alpha\sqrt{N}t]. \end{aligned} \tag{24}$$

The random variables Z_1 and Z_2 have compound Poisson distributions, and can be expressed as

$$Z_1 \stackrel{d.}{=} \sum_{i=1}^{\lfloor t^\epsilon \rfloor} iPoi(Nr(i,t))$$
$$Z_2 \stackrel{d.}{=} \sum_{i=\lfloor t^\epsilon \rfloor+1}^{t} iPoi(Nr(i,t))$$

To derive a bound for Z_1, let $\kappa_o = ND_4t^{2-\epsilon}$, and observe that $N\sum_{i=1}^{\lfloor t^\epsilon \rfloor} i^2 r(i,t) \leq N\sum_{i=1}^{t} i^2 r(i,t) \leq \kappa_o$. Thus, Lemma 4 yields that for $N \geq N_o$ (where N_o is to be specified):

$$\begin{aligned} P[Z_1 - E[Z_1] \geq \alpha t\sqrt{N}] &\leq \exp\left(-\frac{\alpha^2 t^2 N}{2\kappa_o}\psi\left(\frac{\alpha t\sqrt{N}t^{1-\epsilon}}{\kappa_o}\right)\right) \\ &\leq \exp\left(\frac{-\alpha^2 t^\epsilon}{2D_4}\psi\left(\frac{\alpha}{D_4\sqrt{N_o}}\right)\right) \end{aligned} \tag{25}$$

To derive a bound for Z_2 select a constant γ with $\gamma \geq e$. Two cases will be considered. The first case is that $\alpha\sqrt{N} \leq \gamma ND_3t^{-\epsilon}$, or equivalently, that $\sqrt{N} \geq \alpha t^\epsilon/(\gamma D_3)$. Inequality (22) shows that the mean number of connections that contribute at least one packet to Z_2 is less than or equal to $ND_3t^{-\epsilon}$. Therefore, the log moment generating function for $Z_2 - E[Z_2]$ is less than or equal to the log moment generating function of t times a Poisson random variable with mean $ND_3t^{-\epsilon}$. Hence, for $\sqrt{N} \geq \alpha t^\epsilon/(\gamma D_3)$,

$$\begin{aligned} P[Z_2 - E[Z_2] \geq \alpha t\sqrt{N}] &\leq \text{the Chernoff bound for } P[Poi(ND_3t^{-\epsilon}) - ND_3t^{-\epsilon} \geq \alpha\sqrt{N}] \\ &= \exp\left(-\frac{\alpha^2 N}{2ND_3t^{-\epsilon}}\psi\left(\frac{\alpha\sqrt{N}}{ND_3t^{-\epsilon}}\right)\right) \\ &\leq \exp\left(-\frac{\alpha^2 t^\epsilon}{2D_3}\psi(\gamma)\right) \end{aligned} \tag{26}$$

The second case is if $\alpha\sqrt{N} \geq \gamma ND_3t^{-\epsilon}$ and $N \geq N_o$, where the constant N_o is yet to be determined. The following lemma is based on the well known idea of bounding the tail of a Poisson distribution by a geometric series.

Lemma 5. *Let V be a Poisson random variable with mean μ and let c be a constant such that $c \geq \mu e$ and $c \geq 1$. Then $P[V \geq c] \leq \left(\frac{\mu e}{c}\right)^c$.*

Proof. If $\mu e \leq c$ the lemma is trivial, so suppose $\mu e < c$. Then,

$$P[V \geq c] = \sum_{i=\lceil c\rceil}^{\infty} \frac{\exp(-\mu)\mu^i}{i!} \leq \frac{\exp(-\mu)\mu^{\lceil c\rceil}}{\lceil c\rceil!} \sum_{i=\lceil c\rceil}^{\infty} (\frac{\mu}{c})^{i-\lceil c\rceil} = \frac{c}{c-\mu} \frac{\exp(-\mu)\mu^{\lceil c\rceil}}{\lceil c\rceil!}.$$

But $\exp(-\mu) \leq 1$, by Sterlings formula, $\lceil c\rceil! \geq (\lceil c\rceil/e)^{\lceil c\rceil}\sqrt{2\pi}$, and $\frac{c}{c-\mu} \leq \frac{e}{e-1} \leq \sqrt{2\pi}$. So
$P[V \geq c] \leq \left(\frac{\mu e}{\lceil c\rceil}\right)^{\lceil c\rceil} \leq \left(\frac{\mu e}{c}\right)^c$. □

The variable Z_2 is less than what it would be if additional packets were counted so that every type 2 connection generated t packets. Using again that the mean number of such connections is less than or equal to $\lambda = ND_3t^{-\epsilon}$, and using Lemma 5 yields

$$\begin{aligned} P[Z_2 - E[Z_2] \geq \alpha t\sqrt{N}] &\leq P[Z_2 \geq \alpha t\sqrt{N}] \\ &\leq P[Poi(ND_3t^{-\epsilon}) \geq \alpha\sqrt{N}] \\ &\leq \left(\frac{\sqrt{N}D_3t^{-\epsilon}e}{\alpha}\right)^{\alpha\sqrt{N}} \end{aligned} \tag{27}$$

The logarithm of the right side of (27) is a convex function of $\sqrt{N}$, so it is maximized over the relevant range of $\sqrt{N}$, namely $\sqrt{N_o} \leq \sqrt{N} \leq \alpha t^\epsilon/(\gamma D_3)$, at one of the endpoints. Thus, for N in this range,

$$P[Z_2 - E[Z_2] \geq \alpha t\sqrt{N}] \leq \max\left\{(\frac{e}{\gamma})^{-\alpha^2 t^\epsilon/\gamma D_3}, \left(\frac{\sqrt{N_o}D_3t^{-\epsilon}e}{\alpha}\right)^{\alpha\sqrt{N_o}}\right\} \tag{28}$$

Patching together the bound (26) for the first case and the bound (28) for the second case yields that for all $N \geq N_o$ and $T \geq T_o$,

$$P[Z_2-E[Z_2] \geq \alpha t\sqrt{N}] \leq \max\left\{\exp\left(-\frac{\alpha^2 t^\epsilon}{2D_3}\psi(\gamma)\right), (\frac{e}{\gamma})^{-\alpha^2 t^\epsilon/\gamma D_3}, \left(\frac{\sqrt{N_o}D_3t^{-\epsilon}e}{\alpha}\right)^{\alpha\sqrt{N_o}}\right\} \tag{29}$$

To insure that the right side of (29) is summable in t, we require N_o to be large enough that $\epsilon\alpha\sqrt{N_o} > 1$. Combining (24), (25), and (29) completes the proof of Lemma 2.

The second step of the proof of (17) is to bound the fluctuations of X^N within one frame by using a uniform exponential bound on discrete versions of the Brownian bridge, similar to the exact distribution for $\mathcal{M}(\eta^j)$ used in the proof of Theorem 4. Given an integer $j \geq 0$, and $N \geq 1$, let $\eta^{N,j}$ denote the bridge process, defined in terms of X^N the same way the Brownian bridge $\eta^{N,j}$ was defined in terms of X, namely $\eta^{N,j}(t) = X^N(t+j) - ((1-t)X^N_j + tX^N_{j+1})$ for $0 \le t \le 1$.

Lemma 6. *Fix a constant $s_o > 0$. If N is large enough that $N\lambda_N\overline{L} \ge s_o$, then*

$$P[\mathcal{M}(\eta^{N,j}) \ge c] \le 4\exp(-2c^2/(1+12c/\sqrt{s_0}))$$

Proof. The process $\eta^{N,j}$ has the same distribution as a standard Poisson bridge process (for the same mean number of points, $s = N\lambda_N\overline{L}$) on the interval $[0,1]$, sampled at the times of the form j/M, $0 \le j \le M$. The process in between such points is nonincreasing, so that $\eta^{N,j}$ is maximized at a point of the form j/M. Thus, $\mathcal{M}$ is stochastically smaller than the maximum of the standard Poisson bridge. By Inequality 7, p. 575, of [10] (with b there equal to one half) $P[\mathcal{M}(\eta^{N,j}) \ge c] \le 4\exp(-2c^2\psi(4c/\sqrt{s}))$, where s is the mean number of packets arriving in the interval, $s = N\lambda_N\overline{L}$. The proof is completed by applying the bound $\psi(u) \le 1/(1+u/3)$ for $u \ge -1$, also found in [10]. □

Lemmas 2 and 6 and the method of proof of Theorem 4 together imply (17), which completes the proof of Theorem 5.

6. On the distribution of buffer length for the limit process

We haven't found a feasible way of exactly computing the overflow probability $P[Q_0 \ge \beta]$, but lower bounds can be obtained for it. This section presents three different lower bounds, and the most likely path that the process will take to overflow is also described.

6.1. Three lower bounds

This section presents three different lower bounds. Let $A \subset R_+$, then

$$P\left[\sup_{t\ge 0}\{X_t - \theta t\} \ge \beta\right] \ge P\left[\sup_{t\in A}\{X_t - \theta t\} \ge \beta\right] := P^{lb}_\theta(\beta)$$

This lower bound is considered for three different choices of A.

6.1.1. $A = [0,1]$ If $A = [0,1]$ then since X_t is a Wiener process on $[0,1]$, the first lower bound $P_\theta^{lb_1}(\beta)$ is obtained as follows:

$$\begin{aligned} P_\theta^{lb_1}(\beta) &= \int_{-\infty}^{\beta+\theta} P\left[\max_{0\le t\le 1}\{X_t - \theta t\} \ge \beta \mid X_1 = u\right] P[X_1 \in [u, u+du)] \\ &\quad + P\left[X_1 \ge \beta + \theta\right] \\ &\overset{(a)}{=} \int_{-\infty}^{\beta+\theta} \exp\left(-2\beta\left(\beta+\theta-u\right)\right) P[X_1 \in [u,u+du)] \;+\; Q\left(\beta+\theta\right) \\ &= \exp\left(-2\theta\beta\right)\left[1 - Q\left(\theta-\beta\right)\right] \;+\; Q\left(\beta+\theta\right) \end{aligned} \tag{30}$$

where $Q(\cdot)$ denotes the complementary distribution function of a Gaussian random variable with zero mean and unit variance, and (a) follows because conditioning on the right endpoint ($t = 1$) yields a Brownian bridge [2, p. 101], and Equation (13) can be used.

6.1.2. $A = \{t\}$ If $A = \{t\}$ this becomes a well known lower bound (e.g. the *basic approximation* in Addie et al. [1] and Norros [8]) since the random variable $X_t \sim N(0, \rho_t)$ with ρ_t as defined in (6). Hence $P_\theta^{lb_2}(\beta, t) := P\left[X_t - \theta t \ge \beta\right] = Q\left(\frac{\theta t+\beta}{\sqrt{\rho_t}}\right)$. This lower bound can be further refined by taking the supremum over $t \ge 0$ to obtain the second lower bound:

$$P_\theta^{lb_2}(\beta) = \sup_{t\ge 0} P_\theta^{lb_2}(\beta, t) = \begin{cases} 1 & \beta = 0 \\ Q\left(\min\limits_{t\ge 0} \dfrac{\theta t + \beta}{\sqrt{\rho_t}}\right) & \beta > 0 \end{cases} \tag{31}$$

Since ρ_t is a piecewise linear function $P_\theta^{lb_2}(\beta)$ can be easily computed by numerical means.

$P_\theta^{lb_2}(\beta)$ is expected to be tighter than $P_\theta^{lb_1}(\beta)$ for large values of β since in this case overflow is most likely to occur over larger time scales than $t = 1$, making $P_\theta^{lb_1}(\beta)$ a looser bound. Denote by $t^*_{lb_2}$ the optimizing value of t in (31).

6.1.3. $A = [T, T+1]$, *for non-negative integer* T If $A = [T, T+1]$ for non-negative integer T the property presented in Proposition 3, that X is a Brownian bridge between integer time intervals independent of the values of X at the endpoints, becomes very useful.

For any given integer $T \geq 0$, the following lower bound, $P_\theta^{lb_3}(\beta, T)$, can be obtained:

$$
\begin{aligned}
P_\theta^{lb_3}(\beta, T) := & \; P\left[\max_{T \leq t \leq T+1} \{X_t - \theta t\} \geq \beta\right] \\
\overset{(a)}{=} & \; P\left[\max_{0 \leq t \leq 1} \{\eta_t - (\theta + X_T - X_{T+1})\, t\} \geq \beta + \theta T - X_T\right] \\
\overset{(b)}{=} & \; P\left[X_T \geq \beta + \theta T \quad or \quad X_{T+1} \geq \beta + \theta(T+1)\right] \\
& + \int_{-\infty}^{\beta+\theta T} \int_{-\infty}^{\beta+\theta(T+1)} \exp\left(-\frac{2}{\sigma^2}(\beta + \theta T - x_T)(\beta + \theta T - x_T + \theta + x_T - x_{T+1})\right) \\
& \cdot f_X(x_T, x_{T+1}) dx_{T+1} dx_T \\
\overset{(c)}{=} & \; P\left[\hat{X}_T \geq \beta \quad or \quad \hat{X}_{T+1} \geq \beta\right] \\
& + \int_{-\infty}^{\beta} \int_{-\infty}^{\beta} \exp\left(-\frac{2}{\sigma^2}(\beta - x_T)(\beta - x_{T+1})\right) f_{\hat{X}}(x_T, x_{T+1}) dx_{T+1} dx_T \\
= & \; 1 - Q\left(-\frac{\beta - \hat{m}_T}{\rho_T}, -\frac{\beta - \hat{m}_{T+1}}{\rho_{T+1}}, \hat{\rho}\right) \\
& + c_0 \exp(c_1) Q\left(-\frac{\beta - m_1}{\sigma_1}, -\frac{\beta - m_2}{\sigma_2}, \tilde{\rho}\right)
\end{aligned} \tag{32}
$$

where (a) uses the definition of η_t presented in Proposition 3, (b) and (c) follow from (13),
$\begin{pmatrix} X_T \\ X_{T+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \rho_T & \rho_{T,T+1} \\ \rho_{T,T+1} & \rho_{T+1} \end{pmatrix}\right)$ in (b), and
$\begin{pmatrix} \hat{X}_T \\ \hat{X}_{T+1} \end{pmatrix} \sim N\left(\begin{pmatrix} \hat{m}_T \\ \hat{m}_{T+1} \end{pmatrix}, \begin{pmatrix} \rho_T & \rho_{T,T+1} \\ \rho_{T,T+1} & \rho_{T+1} \end{pmatrix}\right)$ in (c). The values of the constants in (32) can be found in Appendix A. Here $Q(x, y, \rho)$ is the two-dimensional Gaussian Q function, and it can be numerically computed as indicated in Appendix B.

This lower bound can be further refined by taking the supremum over nonnegative integers T, to obtain the desired third lower bound:

$$
P_\theta^{lb_3}(\beta) = \sup_{T \geq 0} P_\theta^{lb_3}(\beta, T) \tag{33}
$$

Denote the optimizing value of T in (33) by $T^*_{lb_3}$

It is clear that $P_\theta^{lb_3}(\beta)$ is always tighter than (i.e. greater than) both $P_\theta^{lb_1}(\beta)$ and $P_\theta^{lb_2}(\beta)$. This is because $P_\theta^{lb_3}(\beta)$ is the maximum over all integer intervals of the probability X exceeds β during such an interval, whereas $P_\theta^{lb_2}(\beta)$ is the same for the first interval alone, and $P_\theta^{lb_3}(\beta)$ is the maximum of the probability of exceeding β only at a single time.

6.2. Most likely path to overflow

An interesting question concerns the path that the process takes when it overflows. It is shown in Norros [8] that if Z_t is a stationary increments Gaussian process with $E[Z_t] = 0$, $\mathrm{Cov}(Z_s, Z_t) = \rho_{s,t}$ and $Var(Z_t) = \rho_t$, then the most likely path that Z_t will take to overflow a queue of size β with constant service rate θ is given by

$$Z_s^* = -\frac{\beta + \theta t^*}{\rho_{t^*}} \rho_{-t^*,s}$$

where t^* minimizes $\frac{(\beta+\theta t)^2}{\rho_t}$. Notice that this t^* is the same as the optimal time scale $t^*_{lb_2}$ obtained for $P_\theta^{lb_2}(\beta)$. See Norros [8] for more details.

7. Numerical results

This section considers a specific lifetime distribution for the connections in order to obtain some quantitative analysis of the bounds obtained in Section 6.1. Let the lifetime of a connection have a geometric distribution with parameter $1 - \alpha$ for $0 \le \alpha < 1$. Then $F_L^c(l) = \alpha^{l-1}$ for integer $l \ge 1$. Also $\overline{L} = \frac{1}{1-\alpha}$ and $E\left[L^2\right] < \infty$. Hence the process is not long range dependent. The variance function is

$$\rho_t = t + 2\sum_{j=1}^{\infty} \alpha^j (t-j)^+ \tag{34}$$

Exact overflow probabilities are presented for two specific values of this distribution: $\alpha = 0$ and $\alpha = 1$, whereas bounds are obtained for $0 < \alpha < 1$. These three cases are examined in Sections 7.1, 7.2, and 7.3 respectively. Denote by $P_{\alpha,\theta}(\beta)$ the overflow probability for a particular value of α.

7.1. Case $\alpha = 0$

In the case $\alpha = 0$, the process has no periodicity at all since connections depart after one transmission. It is straightforward to see from (34) and (7) that X becomes a simple Wiener process on $t \ge 0$. Therefore[3], the overflow probability is:

$$P_{0,\theta}(\beta) = \exp(-2\theta\beta)$$

7.2. Case $\alpha = 1$

In the case $\alpha = 1$, the process has sample paths with periodic increments since:

$$X_t = \begin{cases} W_t & 0 \leq t \leq 1 \\ X_{t-1} + W_1 & t \geq 1 \end{cases}$$

where W is a Wiener process.

The overflow probability can be obtained noting that:

$$\sup_{t \geq 0} \{X_t - \theta t\} = \begin{cases} \infty & W_1 > \theta \\ \max\limits_{0 \leq t \leq 1} \{W_t - \theta t\} & W_1 \leq \theta \end{cases}$$

This arises from the fact that if $W_1 > \theta$ then the periodic repetition will start above zero and will always increase so the sample path becomes unbounded above. If instead, $W_1 \leq \theta$ then the process will exceed β only if it does so in the first interval. With this in mind this overflow probability can be calculated by following a procedure similar to the one used in (30) to obtain

$$P_{1,\theta}(\beta) = Q(\theta) + \exp(-2\theta\beta)\left[1 - Q(\theta - 2\beta)\right]$$

7.3. General case $0 < \alpha < 1$

This section presents the overflow probability, obtained from simulations, for both the limit process and the original process in case $0 < \alpha < 1$, and compares them to the lower bounds presented in Section 6.1. The probabilities are plotted in Figure 2 (a) with $\theta = 1$, $\alpha = 0.9$ and $\lambda_N = 0.07793$, which for the original system with $N = 16$ phases corresponds to a load of 0.7793. First, it is clear that the limit process is a very good approximation to the $N = 16$ process. As required, the tightest lower bound is $P^{lb_3}_{\alpha,\theta}$. However, $P^{lb_2}_{\alpha,\theta}$ is very close to $P^{lb_3}_{\alpha,\theta}$ for $\beta > 5$ ($B > 17.66$). Notice in Figure 2 (b) that for $\beta > 0.85$ ($B > 3$), the most likely integer time interval for the process to exceed β is after the first interval (i.e. $T^*_{lb_3} \geq 1$). This causes $P^{lb_1}_{\alpha,\theta}$, which corresponds to $T^*_{lb_3} = 0$, to decrease in accuracy. It can also be observed in Figure 2 that, as expected, $P^{lb_1}_{\alpha,\theta}$ is tighter than $P^{lb_2}_{\alpha,\theta}$ when the most likely frame for the process to exceed β is the first frame. While $P^{lb_3}_{\alpha,\theta}$ is the tightest lower bound, it is also the one that requires the most computation since it requires computing the integrals in (32). In this example, the lower bounds are not a good approximation when the actual overflow probability is larger than 0.1, but they are good for smaller overflow probabilities, which are usually the ones that are of interest to system designers.

Figure 2 shows a clear change in the slope of the overflow probability at $\beta = 0.33$ ($B = 1.16$) for $P^{lb_2}_{\alpha,\theta}$ and $\beta = 0.85$ ($B = 3$) for $P^{lb_3}_{\alpha,\theta}$, where it is more likely for the process to exceed β after the first frame. It is discussed by Pazhyannur and Fleming [9] that for small values of average queueing delay, which by Little's law is proportional to the average queue length, delay behaves exponentially with mean λ_1, while for larger values it behaves exponentially but with mean $\lambda_2 \neq \lambda_1$. This "two-scaled exponential distributions" effect, as referred to by Pazhyannur and Fleming [9], is easy to observe in the semi-log plot in Pazhyannur and Fleming [9]. This two-scale behavior is due to correlation, since if the number of buffered packets is small then most likely they all arrived within a frame of separation, and hence there is no correlation between them. However, a larger number of buffered packets is most likely due to packets that arrived over multiple frames where correlation comes into the picture. This is a characteristic of the semi-periodic behavior of the system, which is preserved in the diffusion limit. This two-scale behavior is also observed in the most likely path to overflow considered by Norros [8].

Figure 3 presents the overflow probabilities and optimal time scales for $\theta = 0.2$, $\alpha = 0.9$ and $\lambda_N = 0.095123$, which corresponds to a load of 0.95123 for $N = 16$ phases. Again, it is clear that the limit process is already a very good approximation to the $N = 16$ process. The same pattern is observed for the lower bounds.

The overflow probabilities and optimal time scales for $\theta = 2.5$, $\alpha = 0.9$ and $\lambda_N = 0.064505$, which corresponds to a load of 0.645054 for $N = 32$ phases, are plotted in Figure 4. The same conclusions are reached. A plot for $N = 32$ rather than for $N = 16$ is shown since for $N = 16$ the overflow probability is small even for very small buffer sizes.

7.4. Effect of mean load and correlation

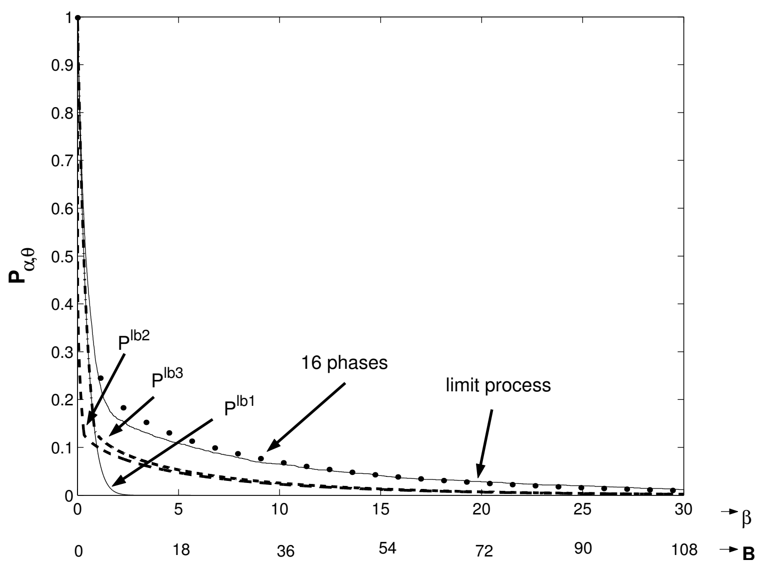
Of course overflow probabilities increase significantly as the mean load increases (i.e., as the draining rate θ decreases). This can be observed in Figure 5, where overflow probabilities below 0.1 are achieved for $\beta > 0.1$ in a lightly loaded system ($\theta = 10$), while they are achieved for $\beta > 1.5$ in a heavily loaded system ($\theta = 1$). The effect of correlation on the overflow probability is a bit more subtle than the effect of mean load. As discussed in Section 7.3, the impact is significant only if overflow probabilities for large enough buffers are considered, and the impact is greater for more heavily loaded systems. Figure 5 indicates a marked dependence on the correlation α only in the more heavily loaded system ($\theta = 1$) and for β sufficiently large ($\beta > 1$).

8. Conclusions

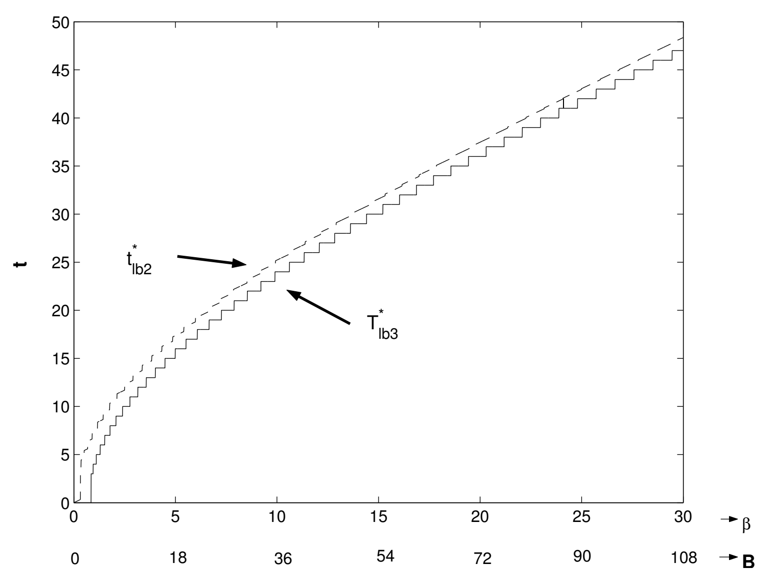
A diffusion limit approximation of the cumulative arrival process in a discrete time queueing system with constant bit rate connections was presented. The diffusion scaling retains the semi-periodic behavior of the process, allowing for short time (within one frame) and long time (multiple frames) analysis in the limit. Several properties of the limit process were discussed. The limit process can be viewed as an interpolation of a stationary increment discrete time Gaussian process, where interpolation is done with Brownian bridges. Under a mild condition on the tail of the connection lifetime distribution, the limit cumulative arrival process satisfies a strong law of large numbers. Related bounds for the actual cumulative arrival process are used to establish the limit in distribution for the scaled equilibrium buffer length.

Bounds on the overflow probability of the limit queueing system as a function of the arrival rate and connection lifetime distribution were presented with some numerical results to evaluate the approximate analysis. It was found that the bounds are a good approximation when the actual probability of overflow is smaller than 0.1. These bounds on the limit system were also compared to the overflow probability of the original system. It was also observed by simulation that the limit approximation to the system is a very good one.

It is also pointed out that the correlation effect is significant only for large buffer build up, and its effect is enhanced when the average load is high.



(a)



(b)

Figure 2: Comparisons for $\alpha = 0.9$, $\theta = 1$, $\lambda_N = 0.07793$: (a) lower bounds; (b) optimal time scale.

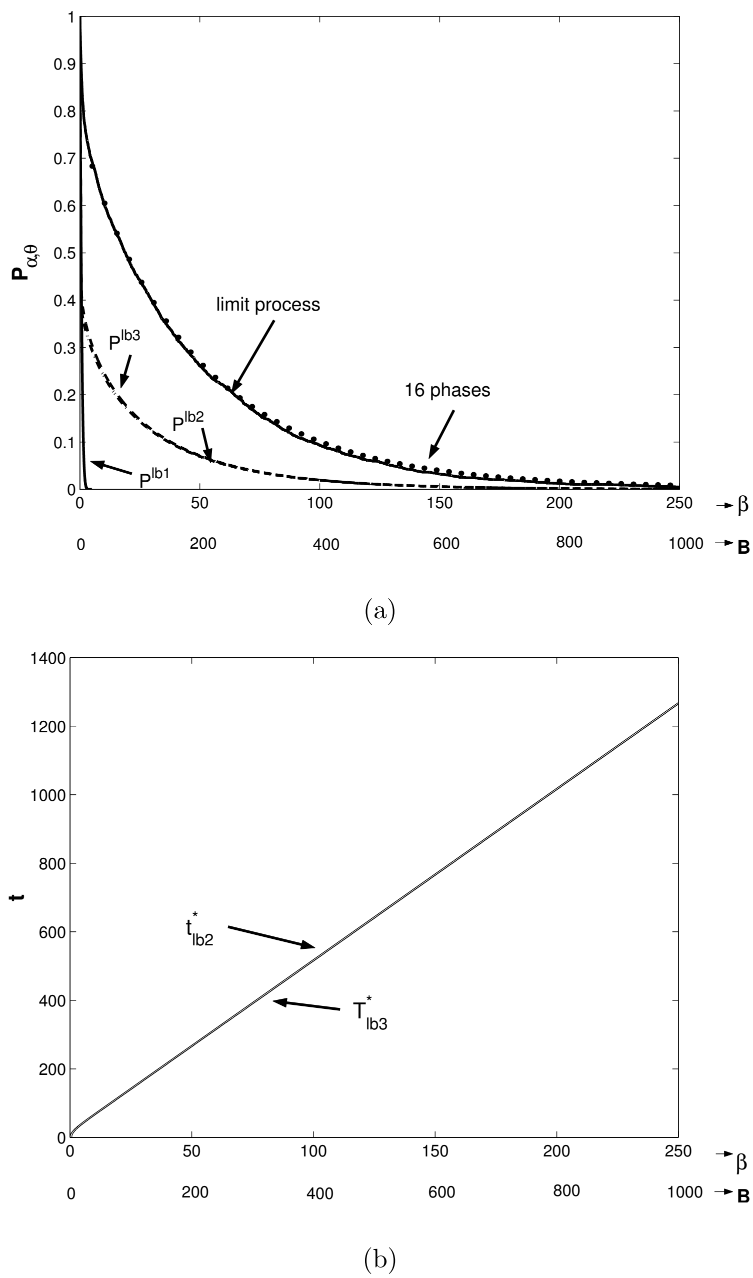


Figure 3: Comparisons for $\alpha = 0.9$, $\theta = 0.2$, $\lambda_N = 0.07793$: (a) lower bounds; (b) optimal time scale.

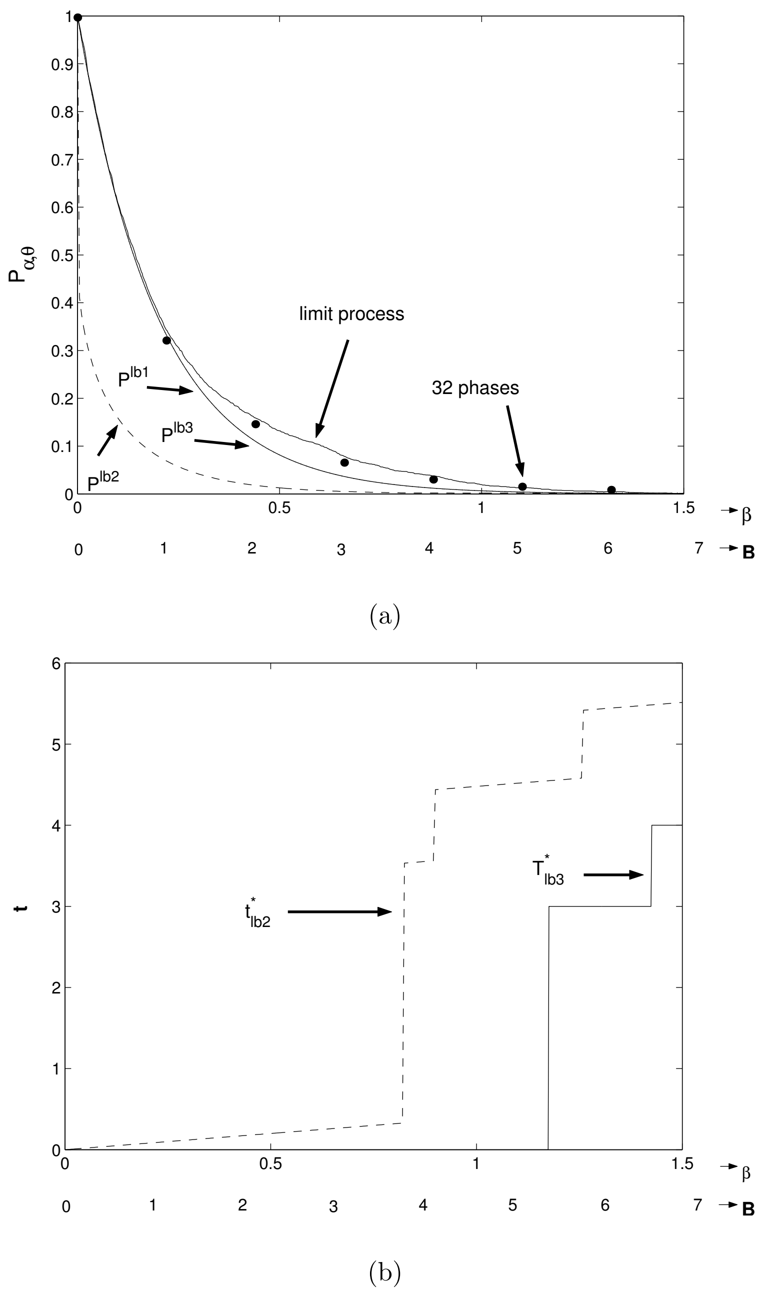
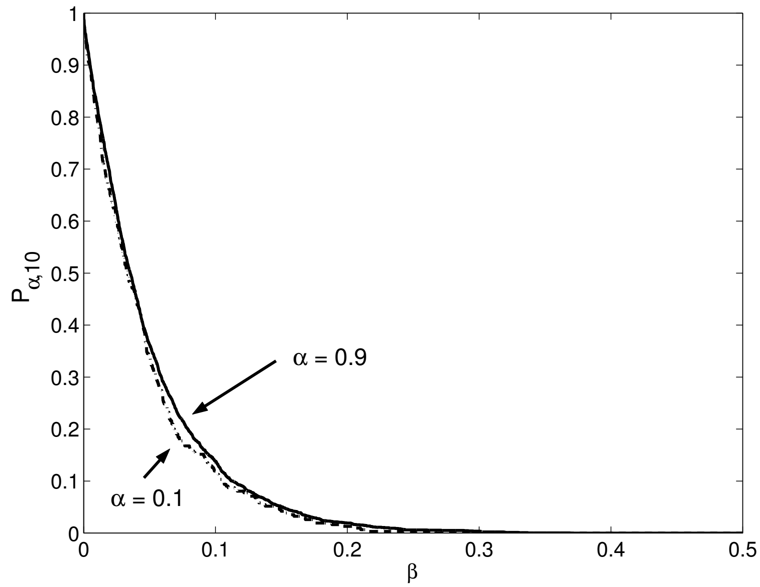
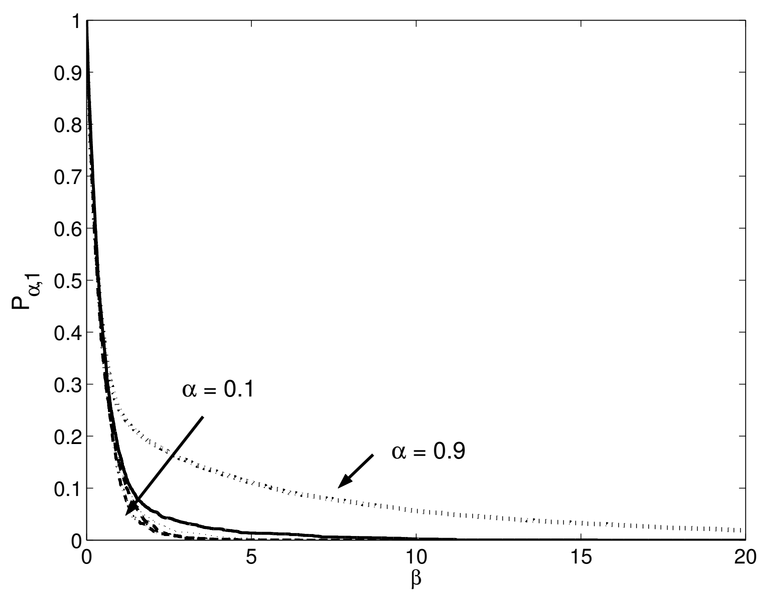


FIGURE 4: Comparisons for $\alpha = 0.9$, $\theta = 2.5$, $\lambda_N = 0.07793$: (a) lower bounds; (b) optimal time scale.



(a)



(b)

FIGURE 5: Overflow probability for: (a) small draining rate ($\theta = 1$); (b) large draining rate ($\theta = 10$).

Appendix A.

This appendix presents the values of the constants that appear in the calculation of $P^{lb_3}_{\alpha,\theta}(\beta)$ in Section 6.1.

$$
\begin{aligned}
\hat{m}_T &= -\theta T \\
\hat{m}_{T+1} &= -\theta(T+1) \\
\hat{\rho} &= \frac{\rho_{T,T+1}}{\sqrt{\rho_T \rho_{T+1}}} \\
\sigma_1^2 &= \frac{\sigma^4 \rho_T}{(\sigma^2 + 2\rho_{T,T+1})^2 - 4\rho_T\rho_{T+1}} \\
\sigma_2^2 &= \frac{\sigma^4 \rho_{T+1}}{(\sigma^2 + 2\rho_{T,T+1})^2 - 4\rho_T\rho_{T+1}} \\
\sigma_{1,2}^2 &= \frac{\sigma^2 \left(\rho_{T,T+1}(\sigma^2 + 2\rho_{T,T+1}) - 2\rho_T\rho_{T+1}\right)}{(\sigma^2 + 2\rho_{T,T+1})^2 - 4\rho_T\rho_{T+1}} \\
\tilde{\rho} &= \frac{\sigma_{1,2}^2}{\sigma_1\sigma_2} = \frac{\rho_{T,T+1}(\sigma^2 + 2\rho_{T,T+1}) - 2\rho_{T,T+1}}{\sigma^2\sqrt{\rho_T\rho_{T+1}}} \\
c_0 &= \frac{\sigma^2}{\sqrt{(\sigma^2 + 2\rho_{T,T+1})^2 - 4\rho_T\rho_{T+1}}} \\
c_1 &= \frac{2\Big((\beta+\theta T)\left((\beta+(T+1)\theta)(\sigma^2 + 2\rho_{T,T+1}) - \rho_{T+1}(\theta T+\beta)\right) - \rho_T(\beta+(T+1)\theta)^2\Big)}{4\rho_T\rho_{T+1} - (\sigma^2 + 2\rho_{T,T+1})^2} \\
m_1 &= \frac{2\rho_T\left(2\beta\rho_{T+1} - \sigma^2(\theta(T+1)+\beta)\right) + (\sigma^2 + 2\rho_{T,T+1})(T\theta\sigma^2 - 2\beta\rho_{T,T+1})}{4\rho_T\rho_{T+1} - (\sigma^2 + 2\rho_{T,T+1})^2} \\
m_2 &= \frac{2\rho_{T+1}\left(2\beta\rho_T - \sigma^2(\theta T+\beta)\right) + (\sigma^2 + 2\rho_{T,T+1})\left((T+1)\theta\sigma^2 - 2\beta\rho_{T,T+1}\right)}{4\rho_T\rho_{T+1} - (\sigma^2 + 2\rho_{T,T+1})^2}
\end{aligned}
$$

Appendix B.

This appendix presents a method to simplify the integration of the two-dimensional Gaussian Q function.

$$
\begin{aligned}
Q(x_1, y_1, \rho) &\overset{def}{=} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{x1}^{\infty}\int_{y1}^{\infty} \exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right) dxdy \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{0}^{\infty}\int_{0}^{\infty} \exp\left(-\frac{(x+x_1)^2+(y+y_1)^2-2\rho(x+x_1)(y+y_1)}{2(1-\rho^2)}\right) dxdy
\end{aligned}
$$

where the equality follows from simple change of variables. This can be further simplified by another change of variables that arises from the geometry of the area of interest. Let

$$N \exp(j\theta) = (x + x_1) + j(y + y_1) \qquad \textit{and} \qquad \phi_s = \arctan\left(\frac{y_1}{x_1}\right)$$

Changing variables from (x, y) to (N, θ) and using the following equality (presented in Simon and Alouini [11]):
$(x + x_1)^2 + (y + y_1)^2 - 2\rho(x + x_1)(y + y_1) = N^2(1 - \rho \sin 2\theta)$, makes it a finite double integral which can be further simplified to single integrals instead of double integrals.

We now present the final integrals for evaluation. The integrals are different depending on which quadrant the vector (x_1, y_1) lies in.

1st quadrant:

$$\begin{aligned} Q(x_1, y_1, \rho) &= \frac{\sqrt{1-\rho^2}}{2\pi} \int_0^{\phi_s} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{y_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\sin^2\theta}\right) d\theta \\ &+ \frac{\sqrt{1-\rho^2}}{2\pi} \int_{\phi_s}^{\pi/2} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{x_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\cos^2\theta}\right) d\theta \end{aligned}$$

2nd quadrant:

$$\begin{aligned} Q(x_1, y_1, \rho) &= \frac{\sqrt{1-\rho^2}}{2\pi} \int_0^{\pi/2} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{y_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\sin^2\theta}\right) d\theta \\ &+ \frac{\sqrt{1-\rho^2}}{2\pi} \int_{-\pi/2}^{\phi_s} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{y_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\sin^2\theta}\right) d\theta \\ &- \frac{\sqrt{1-\rho^2}}{2\pi} \int_{-\pi/2}^{\phi_s} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{x_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\cos^2\theta}\right) d\theta \end{aligned}$$

3rd quadrant:

$$\begin{aligned} Q(x_1, y_1, \rho) &= 1 - \frac{\sqrt{1-\rho^2}}{2\pi} \int_{\phi_s}^{\pi/2} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{y_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\sin^2\theta}\right) d\theta \\ &- \frac{\sqrt{1-\rho^2}}{2\pi} \int_0^{\phi_s} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{x_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\cos^2\theta}\right) d\theta \\ &- \frac{\sqrt{1-\rho^2}}{2\pi} \int_{-\pi/2}^{0} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{y_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\sin^2\theta}\right) d\theta \\ &- \frac{\sqrt{1-\rho^2}}{2\pi} \int_{-\pi/2}^{0} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{x_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\cos^2\theta}\right) d\theta \end{aligned}$$

4th quadrant:

$$\begin{aligned} Q(x_1, y_1, \rho) &= \frac{\sqrt{1-\rho^2}}{2\pi} \int_{\phi_s}^{\pi/2} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{x_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\cos^2\theta}\right) d\theta \\ &- \frac{\sqrt{1-\rho^2}}{2\pi} \int_{\phi_s}^{0} \frac{1}{1-\rho\sin 2\theta} \exp\left(-\frac{y_1^2(1-\rho\sin 2\theta)}{2(1-\rho^2)\sin^2\theta}\right) d\theta \end{aligned}$$

Acknowledgements

The authors are grateful to a referee who pointed out that weak convergence of the process X^N on finite intervals alone is not enough to conclude convergence of the normalized buffer length variable.

This research was supported by the National Science Foundation under grant NSF ANR 99-80544.

References

[1] Addie, R., Mannersalo P. and Norros, I. (1999). Performance formulae for queues with Gaussian input. *Proceedings of the 16th International Teletraffic Congress*, 1169–1178.

[2] Billingsley, P. (1999). *Convergence of probability measures*, 2nd edn. John Wiley & Sons, New York.

[3] Doob, J.L. (1949). Heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statist.* **20**, 393–403.

[4] Durrett, R. (1995). *Probability: Theory and Examples*, 2nd edn. Duxbury Press.

[5] Hajek, B. (1994). A queue with periodic arrivals and constant service rate. In *Probability, Statistics and Optimization—a Tribute to Peter Whittle.* ed. F.P. Kelly. John Wiley & Sons, Chichester, pp. 147–157.

[6] Karatzas, I. and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd edn. Springer.

[7] Krunz, M.M. and Makowski, A.M. (1998). Modeling video traffic using $M/G/\infty$ input processes: A compromise between Markovian and LRD models. *IEEE Journal on Selected Areas in Communications* **16 No.5**, 733–748.

[8] Norros, I. (2002). Most probable path techniques for Gaussian queueing systems. *Networking 2002: Proceedings of the Second International IFIP-TC6 Networking Conference*, 86–104.

[9] Pazhyannur, R.S. and Fleming, P.J. (2001). Asymptotic results for Voice Delay in Packet Networks. *Proceedings of the IEEE Vehicular Technology Conference* **3**, 1861–1865.

[10] Shorack, G.R. and Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics*. John Wiley & Sons, New York.

[11] Simon, M.K. and Alouini, M. (2000). *Digital communication over fading channels : a unified approach to performance analysis* . John Wiley & Sons, New York.